2.4 Oriented manifolds

The compatibility condition between charts (U,φ) and (V,ψ) on a set M is that the map $\psi\circ\varphi^{-1}:\ \varphi(U\cap V)\to\psi(U\cap V)$ is a diffeomorphism. In particular, the Jacobian matrix $D(\psi\circ\varphi^{-1})$ of the transition map is invertible, and hence has non-zero determinant. If the determinant is >0 everywhere, then we say $(U,\varphi),(V,\psi)$ are *oriented-compatible*. An *oriented atlas* on M is an atlas such that any two of its charts are oriented-compatible; a *maximal oriented atlas* is one that contains every chart that is oriented-compatible with all charts in this atlas. An *oriented manifold* is a set with a maximal oriented atlas, satisfying the Hausdorff and countability conditions as in definition 2.7. A manifold is called *orientable* if it admits an oriented atlas.

The notion of an orientation on a manifold will become crucial later, since integration of differential forms over manifolds is only defined if the manifold is oriented.

Example 2.8. The spheres S^n are orientable. To see this, consider the atlas with the two charts (U_+,φ_+) and (U_-,φ_-), given by stereographic projections. (Section 2.3.1.) Here $\varphi_-(U_+\cap U_-)=\varphi_+(U_+\cap U_-)=\mathbb{R}^n\backslash\{0\}$, with transition map $\varphi_-\circ\varphi_+^{-1}(u)=u/||u||^2$. The Jacobian matrix $D(\varphi_-\circ\varphi_+^{-1})(u)$ has entries

$$\Big(D(\varphi_-\circ\varphi_+^{-1})(u)\Big)_{ij}=\frac{\partial}{\partial u^j}\Big(\frac{u^i}{||u||^2}\Big)=\frac{1}{||u||^2}\delta_{ij}-\frac{2u_iu_j}{||u||^4}. \tag{2.2}$$

Its determinant is $-||u||^{-2n}$ (see exercise below).[‖] Hence, the given atlas is *not* an oriented atlas. But this is easily remedied: Simply compose one of the charts, say U_-, with the map $(u^1,u^2,\ldots,u^n)\mapsto(-u^1,u^2,\ldots,u^n)$; then with the resulting new coordinate map $\widetilde{\varphi_-}$ the atlas $(U_+,\varphi_+),(U_-,\widetilde{\varphi_-})$ will be an oriented atlas.

Exercise 13. Calculate the determinant of the matrix with entries (2.2).
(Hint: Check that u is an eigenvector of the matrix, as is any vector orthogonal to u. Alternatively, use that the Jacobian determinant must be invariant under rotations in n-space.)

Example 2.9. We will show in Chapter 6 that $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd or $n=0$. More generally, $\mathrm{Gr}(k,n)$ is orientable if and only if n is even or $n=1$. The complex projective spaces $\mathbb{C}\mathrm{P}^n$ and complex Grassmannians $\mathrm{Gr}_{\mathbb{C}}(k,n)$ are all orientable. This follows because the transition maps for their standard charts, as maps between open subsets of $\mathbb{C}^m$, are actually complex-holomorphic, and this implies that as real maps, their Jacobian has positive determinant. See the following exercise.

[‖] Actually, to decide the sign of the determinant, one does not have to compute the determinant everywhere. If $n>1$, since $\mathbb{R}^n\backslash\{0\}$, it suffices to compute the determinant at just one point, e.g. $u=(1,0,\ldots,0)$.

Exercise 14. Let $A \in \mathrm{Mat}_{\mathbb{C}}(n)$ be a complex $n \times n$-matrix, and $A_{\mathbb{R}} \in \mathrm{Mat}_{\mathbb{R}}(2n)$ the same matrix regarded as a real-linear transformation of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Show that

$$\det_{\mathbb{R}}(A_{\mathbb{R}}) = |\det_{\mathbb{C}}(A)|^2.$$

(Hint: You may want to start with the case $n = 1$, and next consider the case that A is upper triangular.)

2.5 Open subsets

Let M be a set equipped with an m-dimensional maximal atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$.

Definition 2.8. *A subset $U \subseteq M$ is* open *if and only if for all charts $(U_\alpha, \varphi_\alpha) \in \mathscr{A}$ the set $\varphi_\alpha(U \cap U_\alpha)$ is open.*

To check that a subset U is open, it is not actually necessary to verify this condition for all charts. As the following proposition shows, it is enough to check for any collection of charts whose union contains U. In particular, we may take $\mathscr{A}$ in definition 2.8 to be any atlas, not necessarily a maximal atlas.

Proposition 2.1. *Given $U \subseteq M$, let $\mathscr{B} \subseteq \mathscr{A}$ be any collection of charts whose union contains U. Then U is open if and only if for all charts (U_β, φ_β) from $\mathscr{B}$, the sets $\varphi_\beta(U \cap U_\beta)$ are open.*

Proof. In what follows, we reserve the index β to indicate charts (U_β, φ_β) from $\mathscr{B}$. Suppose $\varphi_\beta(U \cap U_\beta)$ is open for all such β. Let $(U_\alpha, \varphi_\alpha)$ be a given chart in the maximal atlas $\mathscr{A}$. We have that

$$\begin{aligned}\varphi_\alpha(U \cap U_\alpha) &= \bigcup_\beta \varphi_\alpha(U \cap U_\alpha \cap U_\beta)\\ &= \bigcup_\beta (\varphi_\alpha \circ \varphi_\beta^{-1})\big(\varphi_\beta(U \cap U_\alpha \cap U_\beta)\big)\\ &= \bigcup_\beta (\varphi_\alpha \circ \varphi_\beta^{-1})\big(\varphi_\beta(U_\alpha \cap U_\beta) \cap \varphi_\beta(U \cap U_\beta)\big).\end{aligned}$$

Since $\mathscr{B} \subseteq \mathscr{A}$, all $\varphi_\beta(U_\alpha \cap U_\beta)$ are open. Hence the intersection with $\varphi_\beta(U \cap U_\beta)$ is open, and so is the pre-image under the diffeomorphism $\varphi_\alpha \circ \varphi_\beta^{-1}$. Finally, we use that a union of open sets is again open. This proves the 'if' part; the 'only if' part is obvious. □

If $\mathscr{A}$ is an atlas on M, and $U \subseteq M$ is open, then U inherits an atlas by restriction:

$$\mathscr{A}_U = \{(U \cap U_\alpha, \varphi_\alpha|_{U \cap U_\alpha})\}.$$

Exercise 15. Verify that if $\mathscr{A}$ is a maximal atlas, then so is $\mathscr{A}_U$, and if this maximal atlas $\mathscr{A}$ satisfies the countability and Hausdorff properties, then so does $\mathscr{A}_U$.

This then proves:

Proposition 2.2. *An open subset of a manifold is again a manifold.*

The collection of open sets of M with respect to an atlas has properties similar to those for $\mathbb{R}^n$:

Proposition 2.3. *Let M be a set with an m-dimensional maximal atlas. The collection of all open subsets of M has the following properties:*

- *$\emptyset, M$ are open.*
- *The intersection $U \cap U'$ of any two open sets U, U' is again open.*
- *The union $\bigcup_i U_i$ of an arbitrary collection $U_i,\ i \in I$ of open sets is again open.*

Proof. All of these properties follow from similar properties of open subsets in $\mathbb{R}^m$. For instance, if U, U' are open, then

$$\varphi_\alpha((U \cap U') \cap U_\alpha) = \varphi_\alpha(U \cap U_\alpha) \cap \varphi_\alpha(U' \cap U_\alpha)$$

is an intersection of open subsets of $\mathbb{R}^m$, hence it is open and therefore $U \cap U'$ is open. □

These properties mean, by definition, that the collection of open subsets of M define a *topology* on M. This allows us to adopt various notions from topology:

1. A subset $A \subseteq M$ is called *closed* if its complement $M \backslash A$ is open.
2. M is called *connected* if the only subsets $A \subseteq M$ that are both closed and open are $A = \emptyset$ and $A = M$.
3. If U is an open subset and $p \in U$, then U is called an *open neighborhood of p*. More generally, if $A \subseteq U$ is a subset contained in M, then U is called an *open neighborhood of A*.

The Hausdorff condition in the definition of manifolds can now be restated as the condition that *any two distinct points p, q in M have disjoint open neighborhoods*. (It is not necessary to take them to be domains of coordinate charts.)

It is immediate from the definition that domains of coordinate charts are open. Indeed, this gives an alternative way of defining the open sets:

Exercise 16. Let M be a set with a maximal atlas. Show that a subset $U \subseteq M$ is open if and only if it is either empty, or is a union $U = \bigcup_{i \in I} U_i$ where the U_i are domains of coordinate charts.

2.6 Compact subsets

Another important concept from topology that we will need is the notion of *compactness*. Recall (e.g. Munkres, Chapter 1 § 4) that a subset $A \subseteq \mathbb{R}^m$ is *compact* if it has the following property: For every collection $\{U_\alpha\}$ of open subsets of $\mathbb{R}^m$ whose union contains A, the set A is already covered by finitely many subsets from that collection. One then proves the important result (see Munkres, Theorems 4.2 and 4.9)

Theorem 2.2 (Heine-Borel). *A subset $A \subseteq \mathbb{R}^m$ is compact if and only if it is closed and bounded.*

While 'closed and bounded' is a simpler characterization of compactness to work with, it does not directly generalize to manifolds (or other topological spaces), while the original definition does:

Definition 2.9. *Let M be a manifold.*[**] *A subset $A \subseteq M$ is* compact *if it has the following property: For every collection $\{U_\alpha\}$ of open subsets of M whose union contains A, the set A is already covered by finitely many subsets from that collection.*

In short, $A \subseteq M$ is compact if every open cover admits a finite subcover.

Proposition 2.4. *If $A \subseteq M$ is contained in the domain of a coordinate chart (U, φ), then A is compact in M if and only if $\varphi(A)$ is compact in $\mathbb{R}^n$.*

Proof. Suppose $\varphi(A)$ is compact. Let $\{U_\alpha\}$ be an open cover of A. Taking intersections with U, it is still an open cover (since $A \subseteq U$). Hence

$$A \subseteq \bigcup_\alpha (U \cap U_\alpha),$$

and therefore

$$\varphi(A) \subseteq \bigcup_\alpha \varphi(U \cap U_\alpha).$$

Since $\varphi(A)$ is compact, there are indices $\alpha_1, \ldots, \alpha_N$ such that

$$\varphi(A) \subseteq \varphi(U \cap U_{\alpha_1}) \cup \ldots \cup \varphi(U \cap U_{\alpha_N}).$$

But then

$$A \subseteq (U \cap U_{\alpha_1}) \cup \ldots \cup (U \cap U_{\alpha_N}) \subseteq U_{\alpha_1} \cup \ldots \cup U_{\alpha_N}.$$

The converse is proved similarly. □

Exercise: Complete the proof, by working out the details for the other direction.

The proposition is useful, since we can check compactness of $\varphi(A)$ by using the Heine-Borel criterion. For more general subsets of M, we can often decide compactness by combining this result with the following:

[**] More generally, the same definition is used for arbitrary topological spaces – e.g., sets with an atlas.

Proposition 2.5. *If $A_1,\dots,A_k \subseteq M$ is a finite collection of compact subsets, then their union $A = A_1 \cup \dots \cup A_k$ is again compact.*

Proof. If $\{U_\alpha\}$ is an open cover of A, then in particular it is an open cover of each of the sets $A_1,\dots,A_k$. For each A_i, we can choose a finite subcover. The collection of all U_α's such that appear in at least one of these subcovers, for $i = 1,\dots k$ are then a finite subcover for A.

Example 2.10. Let $M = S^n$. The closed upper hemisphere $\{x \in S^n |\ x^0 \geq 0\}$ is compact, because is contained in the coordinate chart (U_+, φ_+) for stereographic projection, and its image under φ_+ is the closed and bounded subset $\{u \in \mathbb{R}^n |\ ||u|| \leq 1\}$. Likewise the closed lower hemisphere is compact, and hence S^n itself (as the union of upper and lower hemispheres) is compact.

Example 2.11. Let $\{(U_i,\varphi_i)|\ i = 0,\dots,n\}$ be the standard atlas for $\mathbb{R}\mathrm{P}^n$. Let

$$A_i = \{(x^0 : \dots : x^n) \in \mathbb{R}\mathrm{P}^n |\ ||x||^2 \leq (n+1)\,x_i^2\}.$$

Then $A_i \subseteq U_i$ (since necessarily $x^i \neq 0$ for elements of A_i). Furthermore, $\bigcup_{i=0}^n A_i = \mathbb{R}\mathrm{P}^n$: Indeed, given any $(x^0 : \cdots : x^n) \in \mathbb{R}\mathrm{P}^n$, let i be an index for which $|x^i|$ is maximal. Then $||x||^2 \leq (n+1)x_i^2$ (since the right hand side is obtained from the left hand side by replacing each $(x^j)^2$ with $(x^i)^2 \geq (x^j)^2$)), hence $(x^0 : \cdots : x^n) \in A_i$. Finally, one checks that $\varphi_i(A_i) \subseteq \mathbb{R}^n$ is a closed ball of radius $\sqrt{n+1}$, and in particular is compact.

In a similar way, one can prove compactness of $\mathbb{C}\mathrm{P}^n$, $\mathrm{Gr}(k,n)$, $\mathrm{Gr}_{\mathbb{C}}(k,n)$. However, soon we will have a simpler way of verifying compactness, by showing that they are closed and bounded subsets of $\mathbb{R}^N$ for a suitable N.

Proposition 2.6. *Let M be a set with a maximal atlas. If $A \subseteq M$ is compact, and $C \subseteq M$ is closed, then $A \cap C$ is compact.*

Proof. Let $\{U_\alpha\}$ be an open cover of $A \cap C$. Together with the open subset $M \backslash C$, these cover A. Since A is compact, there are finitely many indices $\alpha_1,\dots,\alpha_N$ with

$$A \subseteq (M \backslash C) \cup U_{\alpha_1} \cup \dots \cup U_{\alpha_N}.$$

Hence $A \cap C \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_N}$. □

The following fact uses the Hausdorff property (and holds in fact for any Hausdorff topological space).

Proposition 2.7. *If M is a manifold, then every compact subset $A \subseteq M$ is closed.*

Proof. Suppose $A \subseteq M$ is compact. Let $p \in M \backslash A$ be given. For any $q \in A$, there are disjoint open neighborhoods V_q of q and U_q of p. The collection of all V_q for $q \in A$ are an open cover of A, hence there exists a finite subcover $V_{q_1},\dots,V_{q_k}$. The intersection $U = U_{q_1} \cap \dots \cap U_{q_k}$ is an open subset of M with $p \in M$ and not meeting $V_{q_1} \cup \dots \cup V_{q_k}$, hence not meeting A. We have thus shown that every $p \in M \backslash A$ has an open neighborhood $U \subseteq M \backslash A$. The union over all such open neighborhoods for all $p \in M \backslash A$ is all of $M \backslash A$, which hence is open. It follows that A is closed. □

Exercise: Let M be the non-Hausdorff manifold from Example 2.6. Find a compact subset $A \subseteq M$ that is not closed.

2.7 Building New Manifolds

2.7.1 Disjoint Union

Given manifolds M, M' of the same dimensions m, with atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U'_\beta, \varphi'_\beta)\}$, the disjoint union $N = M \sqcup M'$ is again an m-dimensional manifold with atlas $\{U_\alpha, \varphi_\alpha)\} \cup \{U'_\beta, \varphi'_\beta)\}$. This manifold N is not much more interesting than considering M and M' separately, but is the first step towards "gluing" M and M' in an interesting way, which often results in genuinely new manifold (more below).

2.7.2 Products

Given manifolds M, M' of dimensions m, m' (not necessarily the same), with atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U'_\beta, \varphi'_\beta)\}$, the cartesian product $M \times M'$ is a manifold of dimension $m + m'$. An atlas is given by the product charts $U_\alpha \times U'_\beta$ with the product maps $\varphi_\alpha \times \varphi'_\beta : \ (x, x') \mapsto (\varphi_\alpha(x), \varphi'_\beta(x'))$. For example, the 2-torus $T^2 = S^1 \times S^1$ becomes a manifold in this way, and likewise the n-torus

$$T^n = S^1 \times \cdots \times S^1.$$

Question: One may repeat both of the constructions above recursively, a finite number of times. Recall that the countable union of countable sets is countable (see Appendix). Can we therefore repeat this construction a countable number of times?

2.7.3 Quotients

When we constructed $\mathbb{R}\mathrm{P}^n$, we did so by defining an equivalence relation on $\mathbb{R}^n$. On the other hand, the double-origin non-example 2.6 was also defined by considering an equivalence relation on $\mathbb{R} \sqcup \mathbb{R}$. In one case the result was a manifold, and in the other it was not. The non-example actually shows us what may fail: the quotient may fail to be Hausdorff. We will revisit quotient construction at the end of Chapter 3, after we have developed more tools. The remainder of the section is aimed toward students with background in basic point-set topology.

> **Exercise 17.** Let T be a Hausdorff topological space. Let R be an equivalence relation on T, and $T' = T/R$ the set of equivalence classes. There is a natural quotient map $\pi : T \to T'$ taking each element to its equivalence class.
> Recall that the quotient topology on T' is defined as follows: $U' \subseteq T'$ is open iff $\pi^{-1}(U')$ is open in T.
> Suppose π is an open map (that is $\pi(U)$ is open in T' for any open set U of T).
> Show that if $R \subseteq T \times T$ is closed, then T' is Hausdorff.

Suppose $\{U_\alpha\}_{\alpha=1}^{\infty}$ is some countable collection of open sets in $\mathbb{R}^n$. By the previous section, each U_α is an n-manifold. Further suppose that for each two indices α, β we have a diffeomorphism $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to U_\alpha \cap U_\beta$ satisfying the following conditions:

- $\varphi_{\alpha\alpha}$ is the identity map.
- For any three indices α, β, γ we have

$$\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}.$$

Exercise 18. Let $M = \bigsqcup_\alpha U_\alpha$. Define the following relation on M. Let $x, y \in M$ and suppose $x \in U_\alpha$, $y \in U_\beta$ (not necessarily distinct). Say

$$x \sim y \iff \varphi_{\alpha\beta}(x) = y.$$

Show that this is an equivalence relation on M.
Equip $M/\sim$ with the quotient topology. Suppose you knew that $M/\sim$ is Hausdorff; show there is a natural manifold structure on $M/\sim$ making it an n-dimensional manifold.

Every manifold can be constructed as in the exercise above; where the U_i's are simply the chart domains, and the $\varphi_{\alpha\beta}$ are the transition functions. However, not every such construction results in a manifold; once again the problem is the Hausdorff condition, which we assumed in the Exercise above.