2.8 Appendix

2.8.1 Countability

A set X is *countable* if it is either finite (possibly empty), or there exists a bijective map $f:\ \mathbb{N} \to X$. We list some basic facts about countable sets:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable, $\mathbb{R}$ is not countable.
- If X_1, X_2 are countable, then the cartesian product $X_1 \times X_2$ is countable.
- If X is countable, then any subset of X is countable.
- If X is countable, and $f:\ X \to Y$ is surjective, then Y is countable.
- If $(X_i)_{i\in I}$ are countable sets, indexed by a countable set I, then the (disjoint) union $\sqcup_{i\in I} X_i$ is countable.

2.8.2 Equivalence relations

We will make extensive use of *equivalence relations*; hence it may be good to review this briefly. A *relation* from a set X to a set Y is simply a subset

$$R \subseteq Y \times X.$$

We write $x \sim_R y$ if and only if $(y,x) \in R$. When R is understood, we write $x \sim y$. If $Y = X$ we speak of a *relation on* X.

Example 2.12. Any map $f:\ X \to Y$ defines a relation, given by its *graph* $\mathrm{Gr}_f = \{(f(x),x) | x \in X\}$. In this sense relations are generalizations of maps; for example, they are often used to describe 'multi-valued' maps.

Remark 2.10. Given another relation $S \subseteq Z \times Y$, one defines a composition $S \circ R \subseteq Z \times X$, where

$$S \circ R = \{(z,x) |\ \exists y \in Y:\ (z,y) \in S,\ (y,x) \in R\}.$$

Our conventions are set up in such a way that if $f:\ X \to Y$ and $g:\ Y \to Z$ are two maps, then $\mathrm{Gr}_{g\circ f} = \mathrm{Gr}_g \circ \mathrm{Gr}_f$.

Example 2.13. On the set $X = \mathbb{R}$ we have relations $\geq, >, <, \leq, =$. But there is also the relation defined by the condition $x \sim x' \Leftrightarrow x' - x \in \mathbb{Z}$, and many others.

A relation $\sim$ on a set X is called an *equivalence relation* if it has the following properties,

1. Reflexivity: $x \sim x$ for all $x \in X$,
2. Symmetry: $x \sim y \Rightarrow y \sim x$,
3. Transitivity: $x \sim y,\ y \sim z \Rightarrow x \sim z$.

Given an equivalence relation, we define the *equivalence class* of $x \in X$ to be the subset

$$[x] = \{y \in X |\ x \sim y\}.$$

Note that X is a disjoint union of its equivalence classes. We denote by $X/\sim$ the set of equivalence classes. That is, all the elements of a given equivalence class are lumped together and represent a single element of $X/\sim$. One defines the *quotient map*

$$q:\ X \to X/\sim,\quad x \mapsto [x].$$

By definition, the quotient map is surjective.

Remark 2.11. There are two other useful ways to think of equivalence relations:

- An equivalence relation R on X amounts to a decomposition $X = \sqcup_{i\in I} X_i$ as a disjoint union of subsets. Given R, one takes X_i to be the equivalence classes; given the decomposition, one defines $R = \{(y,x) \in X \times X | \exists i \in I:\ x,y \in X_i\}$.
- An equivalence relation amounts to a surjective map $q:\ X \to Y$. Indeed, given R one takes $Y := X/\sim$ with q the quotient map; conversely, given q one defines $R = \{(y,x) \in X \times X|\ q(x) = q(y)\}$.

Remark 2.12. Often, we will not write out the entire equivalence relation. For example, if we say "*the equivalence relation on S^2 given by $x \sim -x$*", then it is understood that we also have $x \sim x$, since reflexivity holds for any equivalence relation. Similarly, when we say "*the equivalence relation on $\mathbb{R}$ generated by $x \sim x+1$*", it is understood that we also have $x \sim x+2$ (by transitivity: $x \sim x+1 \sim x+2$) as well as $x \sim x-1$ (by symmetry), hence $x \sim x+k$ for all $k \in \mathbb{Z}$. (Any relation $R_0 \subseteq X \times X$ extends to a unique smallest equivalence relation R; one says that R is the equivalence relation generated by R_0.)

Example 2.14. Consider the equivalence relation on S^2 given by

$$(x,y,z) \sim (-x,-y,-z).$$

The equivalence classes are pairs of antipodal points; they are in 1-1 correspondence with lines in $\mathbb{R}^3$. That is, the quotient space $S^2/\sim$ is naturally identified with $\mathbb{R}\mathrm{P}^2$.

Example 2.15. The quotient space $\mathbb{R}/\sim$ for the equivalence relation $x \sim x+1$ on $\mathbb{R}$ is naturally identified with S^1. If we think of S^1 as a subset of $\mathbb{R}$, the quotient map is given by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$.

Example 2.16. Similarly, the quotient space for the equivalence relation on $\mathbb{R}^2$ given by $(x,y) \sim (x+k, y+l)$ for $k,l \in \mathbb{Z}$ is the 2-torus T^2.

Example 2.17. Let E be a k-dimensional real vector space. Given two ordered bases $(e_1,\ldots,e_k)$ and $(e'_1,\ldots,e'_k)$, there is a unique invertible linear transformation $A:\ E \to E$ with $A(e_i) = e'_i$. The two ordered bases are called *equivalent* if $\det(A) > 0$. One checks that equivalence of bases is an equivalence relation. There are exactly two equivalence classes; the choice of an equivalence class is called an *orientation* on E. For example, $\mathbb{R}^n$ has a standard orientation defined by the standard basis $(e_1,\ldots,e_n)$. The opposite orientation is defined, for example, by $(-e_1,e_2,\ldots,e_n)$. A permutation of the standard basis vectors defines the standard orientation if and only if the permutation is even.