3

Smooth maps

3.1 Smooth functions on manifolds

A real-valued function on an open subset $U \subseteq \mathbb{R}^n$ is called smooth if it is infinitely differentiable. The notion of smooth functions on open subsets of Euclidean spaces carries over to manifolds: A function is smooth if its expression in local coordinates is smooth.

Definition 3.1. *A function $f:\ M \to \mathbb{R}^n$ on a manifold M is called* smooth *if for all charts (U,φ) the function*

$$f \circ \varphi^{-1}:\ \varphi(U) \to \mathbb{R}^n$$

is smooth. The set of smooth functions into the real-line $f : M \to \mathbb{R}$ is denoted $C^\infty(M)$.

One does not have to actually check every single chart. Since transition maps are diffeomorphisms, it suffices to check the condition for the charts from any given atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$, which need not be the maximal atlas. Indeed, if the condition on f holds for all charts from the atlas $\mathscr{A}$, and if (U,φ) is another chart compatible with $\mathscr{A}$, then the functions

$$f \circ \varphi^{-1}\big|_{\varphi(U\cap U_\alpha)} = (f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi^{-1}):\ \varphi(U \cap U_\alpha) \to \mathbb{R}^n$$

are smooth, and since the open sets $\varphi(U \cap U_\alpha)$ cover $\varphi(U)$ this implies smoothness of $f \circ \varphi^{-1}$.

Smoothness, like continuity, is a local condition: given an open subset $U \subseteq M$, we say that a function f is smooth on U if its restriction $f|_U$ is smooth. (Here we are using that U itself is a manifold.) Given $p \in M$, we say that f is smooth at p if it is smooth on some open neighborhood of p.

Example 3.1. The 'height function'

$$f:\ S^2 \to \mathbb{R},\ \ (x,y,z) \mapsto z$$

is smooth. In fact, we see that for any smooth function $h \in C^\infty(\mathbb{R}^3)$ (for example the coordinate functions), the restriction $f = h|_{S^2}$ is again smooth. This may be checked using the 6-charts atlas given by projection onto the coordinate planes: E.g., in the chart $U = \{(x,y,z) |\, z > 0\}$ with $\varphi(x,y,z) = (x,y)$, we have

$$(f \circ \varphi^{-1})(x,y) = h\left(x, y, \sqrt{1-(x^2+y^2)}\right)$$

which is smooth on $\varphi(U) = \{(x,y) |\, x^2+y^2 < 1\}$. (The argument for the other charts in this atlas is similar.)

Of course, if h is not a smooth function on $\mathbb{R}^3$, it may still happen that its restriction to S^2 is smooth.

Exercise 19. Show that the map

$$f : \mathbb{R}\mathrm{P}^2 \to \mathbb{R},\ (x:y:z) \mapsto \frac{yz+xz+xy}{x^2+y^2+z^2}$$

is well-defined and smooth.

Exercise 20. Prove that the map

$$f :\ S^2 \to \mathbb{R},\ (x,y,z) \mapsto \sqrt{1-z^2}$$

is smooth only on $S^2 \backslash \{(0,0,1),(0,0,-1)\}$. To analyse the situation near the north pole, use the coordinate chart (U,φ) as above. In these coordinates, $z = \sqrt{1-(x^2+y^2)}$, hence $\sqrt{1-z^2} = \sqrt{x^2+y^2}$ which is not smooth near $(x,y) = (0,0)$.

Example 3.2. Let

$$\pi :\ \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}\mathrm{P}^n$$

be the quotient map. Given $f :\ \mathbb{R}\mathrm{P}^n \to \mathbb{R}^m$, the function

$$\widehat{f} = f \circ \pi :\ \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}^m$$

satisfies $\widehat{f}(\lambda x) = \widehat{f}(x)$ for $\lambda \neq 0$; conversely any $\widehat{f}$ with this property descends to a function f on the projective space. We claim that f is smooth if and only if $\widehat{f}$ is smooth. To see this, note that in the standard coordinate chart (U_i, φ_i) for $\mathbb{R}\mathrm{P}^n$, the function φ_i^{-1} may be written as the composition of smooth maps $\pi \circ g$, where

$$g : \mathbb{R}^n \supseteq U_i \to \mathbb{R}^{n+1} \backslash \{0\},\ (u^1, \ldots, u^n) \mapsto (u^1, \ldots, u^i, 1, u^{i+1}, \ldots, u^n).$$

Then, $f \circ \varphi^{-1} = \widehat{f} \circ g$. Hence, if $\widehat{f}$ is smooth then so is f. (The converse is similar.) As a special case, we see that for all $0 \leq j \leq k \leq n$ the functions

$$f :\ \mathbb{R}\mathrm{P}^n \to \mathbb{R},\ (x^0 : \ldots : x^n) \mapsto \frac{x^j x^k}{||x||^2} \tag{3.1}$$

are well-defined and smooth. By a similar argument, the functions

$$f:\ \mathbb{C}\mathrm{P}^n \to \mathbb{C},\ (z^0:\ldots:z^n) \mapsto \frac{\overline{z^j}z^k}{||z||^2} \tag{3.2}$$

(where the bar denotes complex conjugation) are well-defined and smooth, in the sense that both the real and imaginary parts are smooth.

Exercise 21. In the example above we have used the fact that the quotient map

$$\pi : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}\mathrm{P}^n,\ (x^0,\ldots,x^n) \mapsto (x^0:\ldots:x^n)$$

is smooth. Prove this.

Exercise 22. Show once again that the map

$$f : \mathbb{R}\mathrm{P}^2 \to \mathbb{R},\ (x:y:z) \mapsto \frac{yz+xy+xz}{x^2+y^2+z^2}$$

is smooth, this time using the conclusion of Example **??** above.

Lemma 3.1. *Smooth functions* $f : M \to \mathbb{R}^n$ *are* continuous: *For every open subset* $J \subseteq \mathbb{R}^n$*, the pre-image* $f^{-1}(J) \subseteq M$ *is open.*

Proof. We have to show that for every (U,φ), the set $\varphi(U \cap f^{-1}(J)) \subseteq \mathbb{R}^m$ is open. But this subset coincides with the pre-image of J under the map $f \circ \varphi^{-1}:\ \varphi(U) \to \mathbb{R}^n$, which is a smooth function on an open subset of $\mathbb{R}^m$, and these are (by definition) continuous.

Exercise 23. We have characterised smooth functions as functions that are smooth "in charts." There is a similar characterisation for continuous functions: Show that $f:\ M \to \mathbb{R}^n$ *continuous* (i.e., the pre-image of any open subset $J \subseteq \mathbb{R}^n$ under f is open) if and only if for all charts (U,φ) the function $f \circ \varphi^{-1}$ is continuous.

From the properties of smooth functions on $\mathbb{R}^m$, one immediately gets the following properties of smooth functions on manifolds M:

- If $f,g \in C^\infty(M)$ and $\lambda,\mu \in \mathbb{R}$, then $\lambda f + \mu g \in C^\infty(M)$.
- If $f,g \in C^\infty(M)$, then $fg \in C^\infty(M)$.
- $1 \in C^\infty(M)$ (where 1 denotes the constant function $p \mapsto 1$).

These properties say that $C^\infty(M)$ is an *algebra* with unit 1. (See the appendix to this chapter for some background information on algebras.) Below, we will develop many of the concepts of manifolds in terms of this algebra of smooth functions.

Exercise 24. Prove the assertion that $f, g \in C^\infty(M) \implies fg \in C^\infty(M)$.

Suppose M is any set with a maximal atlas $\{(U_\alpha, \varphi_\alpha)\}$. The definition of $C^\infty(M)$ does not use the Hausdorff or countability conditions; hence it makes sense in this more general context. This means that we may use functions to check the Hausdorff property:

Proposition 3.1. *Suppose M is any set with a maximal atlas, and $p \neq q$ are two points in M. Then the following are equivalent:*

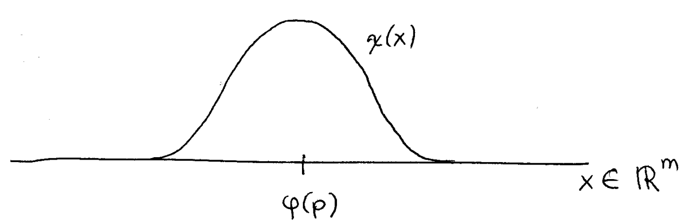
(i) There are open subsets $U, V \subseteq M$ with $p \in U$, $q \in V$, $U \cap V = \emptyset$,
(ii) There exists a continuous $f : M \to \mathbb{R}$ with $f(p) \neq f(q)$.
(iii) There exists $f \in C^\infty(M)$ with $f(p) \neq f(q)$.

Proof. Lemma 3.1 shows that $(iii) \Rightarrow (ii)$. Thus, it suffices to show $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (i)$.

"$(i) \Rightarrow (iii)$". Suppose (i) holds. As explained in Section 2.5, we may take U, V to be the domains of coordinate charts (U, φ) and (V, ψ) around p, q. Choose $\varepsilon > 0$ such that the closed ε-ball

$$\overline{B_\varepsilon(\varphi(p))} = \left\{x \in \mathbb{R}^m \middle| \, ||x - \varphi(p)|| \leq \varepsilon\right\}$$

is contained in $\varphi(U)$; let $A \subseteq U$ be its pre-image under φ. Let $\chi \in C^\infty(\mathbb{R}^m)$ be a 'bump function' centered at $\varphi(p)$, with $\chi(\varphi(p)) = 1$ and $\chi(x) = 0$ for $||x - \varphi(p)|| \geq \varepsilon$. (For the existence of such a function see Munkres, Lemma 16.1, or Lemma A.2 in the appendix.)



The function $f : M \to \mathbb{R}$ such that $f = \chi \circ \varphi$ on U and $f = 0$ on $M \backslash A$ is smooth, and satisfies $f(p) = 1, \ f(q) = 0$.

"$(ii) \Rightarrow (i)$". Suppose (ii) holds. Let $\delta = |f(q) - f(p)|/2$, and put

$$U = \{x \in M | \, |f(x) - f(p)| < \delta\}, \tag{3.3}$$
$$V = \{x \in M | \, |f(x) - f(q)| < \delta\} \tag{3.4}$$

Thus, U, V are the f-preimages of δ-balls centred at $f(p)$ and $f(q)$. Since f is continuous, U, V are open, and clearly $p \in U, \ q \in V, U \cap V = \emptyset$.

A consequence of this result is:

Corollary 3.1 (Criterion for Haudorff condition). *A set M with an atlas satisfies the Hausdorff condition if and only if for any two distinct points $p,q \in M$, there exists a continuous function $f : M \to \mathbb{R}$ with $f(p) \neq f(q)$. In particular, if there exists a continuous injective map $F : M \to \mathbb{R}^N$, then M is Hausdorff.*

Exercise 25. Prove the last assertion: if there exists a smooth injective map $F : M \to \mathbb{R}^N$, then M is Hausdorff.

Example 3.3 (Projective spaces). Write vectors $x \in \mathbb{R}^{n+1}$ as column vectors, hence $x^\top$ is the corresponding row vector. The matrix product $xx^\top$ is a square matrix with entries $x^j x^k$. The map

$$\mathbb{R}\mathrm{P}^n \to \mathrm{Mat}_{\mathbb{R}}(n+1),\ \ (x^0 : \ldots : x^n) \mapsto \frac{x\,x^\top}{||x||^2} \tag{3.5}$$

is a smooth; indeed, its matrix components are the functions (3.1). For any given $(x^0 : \ldots : x^n) \in \mathbb{R}\mathrm{P}^n$, at least one of these components is non-zero. Identifying $\mathrm{Mat}_{\mathbb{R}}(n+1) \cong \mathbb{R}^N$, where $N = (n+1)^2$, this gives the desired smooth injective map from projective space into $\mathbb{R}^N$; hence the criterion applies, and the Hausdorff condition follows. For the complex projective space, one similarly has a smooth and injective map

$$\mathbb{C}\mathrm{P}^n \to \mathrm{Mat}_{\mathbb{C}}(n+1),\ \ (z^0 : \ldots : z^n) \mapsto \frac{z\,z^\dagger}{||z||^2} \tag{3.6}$$

(where $z^\dagger = \overline{z}^\top$ is the conjugate transpose of the complex column vector z) into $\mathrm{Mat}_{\mathbb{C}}(n+1) = \mathbb{R}^N$ with $N = 2(n+1)^2$.

Exercise 26. Verify that the map

$$\mathrm{Gr}(k,n) \to \mathrm{Mat}_{\mathbb{R}}(n),\ \ E \mapsto P_E, \tag{3.7}$$

taking a subspace E to the matrix of the orthogonal projection onto E, is smooth and injective, hence $\mathrm{Gr}(k,n)$ is Hausdorff. Discuss a similar map for the complex Grassmannian $\mathrm{Gr}_{\mathbb{C}}(k,n)$.

In the opposite direction, the criterion tells us that for a set M with an atlas, if the Hausdorff condition does not hold then no smooth injective map into $\mathbb{R}^N$ exists.

Example 3.4. Consider the non-Hausdorff manifold M from Example 2.6. Here, there are two points p,q that do not admit disjoint open neighborhoods, and we see directly that any smooth function on M must take on the same values at p and q: With the coordinate charts $(U,\varphi),(V,\psi)$ in that example,

$$f(p) = f(\varphi^{-1}(0)) = \lim_{t\to 0^-} f(\varphi^{-1}(t)) = \lim_{t\to 0^-} f(\psi^{-1}(t)) = f(\psi^{-1}(0)) = f(q),$$

since $\varphi^{-1}(t) = \psi^{-1}(t)$ for $t < 0$.

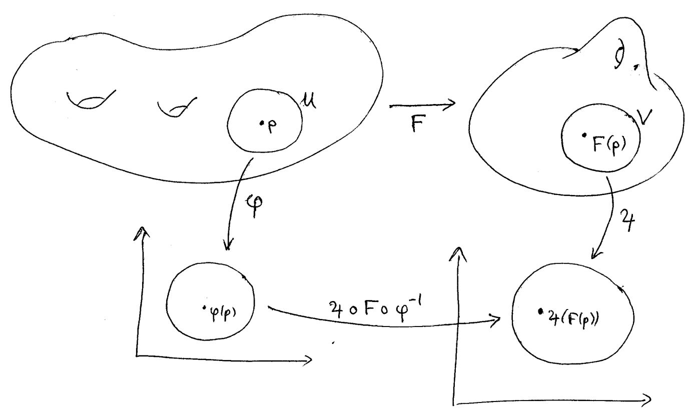
3.2 Smooth maps between manifolds

The notion of smooth maps from M to $\mathbb{R}^n$ generalizes to smooth maps between manifolds.

Definition 3.2. *A map $F: M \to N$ between manifolds is* smooth at $p \in M$ *if there are coordinate charts (U, φ) around p and (V, ψ) around $F(p)$ such that $F(U) \subseteq V$ and such that the composition*

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is smooth. The function F is called a smooth map from M to N *if it is smooth at all $p \in M$.*



As before, to check smoothness of F, it suffices to take *any* atlas $\{(U_\alpha, \varphi_\alpha)\}$ of M with the property that $F(U_\alpha) \subseteq V_\alpha$ for *some* chart (V_α, ψ_α) of N, and then check smoothness of the maps

$$\psi_\alpha \circ F \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \psi_\alpha(V_\alpha).$$

This is because the condition for smoothness at p does not depend on the choice of charts: Given a different choice of charts (U', φ') and (V', ψ') with $F(U') \subseteq V'$, we have

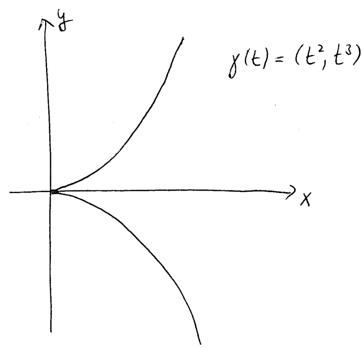
$$\psi' \circ F \circ (\varphi')^{-1} = (\psi' \circ \psi^{-1}) \circ (\psi \circ F \circ (\varphi)^{-1}) \circ (\varphi \circ (\varphi')^{-1})$$

on $\varphi'(U \cap U')$. This demonstrates once again that the requirement that the transition maps be diffeomorphisms is the correct idea.

The collection of smooth maps $f : M \to N$ is denoted $C^\infty(M,N)$. Note that since $\mathbb{R}$ itself is a manifold, we now have two definition for a smooth map $f : M \to \mathbb{R}$. You should convince yourself that the two definitions coincide:

$$C^\infty(M,\mathbb{R}) = C^\infty(M).$$

Smooth functions $\gamma\colon\ J \to M$ from an open interval $J \subseteq \mathbb{R}$ to M are called *(smooth) curves in M*. Note that the image of a smooth curve need not look smooth. For instance, the image of $\gamma\colon\ \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t^2, t^3)$ has a 'cusp singularity' at $(0,0)$.



> **Exercise 27.** Suppose $F \in C^\infty(M,N)$.
>
> 1. Let (U,φ) be a coordinate chart for M and (V,ψ) a coordinate chart for N, with $F(U) \subseteq V$. Show that for all open subsets $W \subseteq N$ the set $U \cap F^{-1}(W)$ is open.
> 2. Given $p \in M$ and *any* chart (V,ψ) around $F(p)$, show that there exists a chart (U,φ) around p such that $F(U) \subseteq V$.

Proposition 3.2. *Smooth maps between manifolds are continuous.*

> **Exercise 28.** Use the previous exercise to prove the proposition above.

Proposition 3.3. *Suppose $F_1 :\ M_1 \to M_2$ and $F_2 :\ M_2 \to M_3$ are smooth maps. Then the composition*

$$F_2 \circ F_1 :\ M_1 \to M_3$$

is smooth.

> **Exercise 29.** Prove the proposition.
> Hint: Use the previous exercise.

3.2.1 Diffeomorphisms of manifolds

Definition 3.3. *A smooth map $F : M \to N$ is called a* diffeomorphism *if it is invertible, with a smooth inverse $F^{-1} : N \to M$. Manifolds M,N are called* diffeomorphic *if there exists a diffeomorphism from M to N.*

In other words, a diffeomorphism of manifolds is a bijection of the underlying sets that identifies the maximal atlases of the manifolds. Manifolds that are diffeomorphic are therefore considered 'the same manifold'.

Similarly, a continuous map $F: M \to N$ is called a *homeomorphism* if it is invertible, with a continuous inverse. Manifolds that are homeomorphic are considered 'the same topologically'. Since every smooth map is continuous, every diffeomorphism is a homeomorphism.

Example 3.5. By definition, every coordinate chart (U, φ) on a manifold M gives a diffeomorphism $\varphi: U \to \varphi(U)$ onto an open subset of $\mathbb{R}^m$.

Example 3.6. The standard example of a homeomorphism of smooth manifolds that is not a diffeomorphism is the map

$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3.$$

Indeed, this map is smooth and invertible, but the inverse map $y \mapsto y^{\frac{1}{3}}$ is not smooth.

Example 3.7. Give a manifold M with maximal atlas $\mathscr{A}$, any homeomorphism $F: M \to M$ can be used to define a new atlas $\mathscr{A}'$ on M, with charts $(U', \varphi') \in \mathscr{A}'$ obtained from charts $(U, \varphi) \in \mathscr{A}$ as $U' = F(U)$, $\varphi' = \varphi \circ F^{-1}$. One can verify (please do) that $\mathscr{A}' = \mathscr{A}$ if and only if F is a diffeomorphism. Thus, if F is a homeomorphism of M which is not a diffeomorphism, then F defines a new atlas $\mathscr{A}' \neq \mathscr{A}$.

However, the new manifold structure on M is not genuinely different from the old one. Indeed, while $F: M \to M$ is not a diffeomorphism relative to the atlas $\mathscr{A}$ on the domain M and target M, it *does* define a diffeomorphism if we use the atlas $\mathscr{A}$ on the domain and the atlas $\mathscr{A}'$ on the target. Hence, even though $\mathscr{A}$ and $\mathscr{A}'$ are different atlases, the resulting manifold structures are still diffeomorphic.

Exercise 30. Consider $\mathbb{R}$ with the trivial atlas $\mathscr{A} := (\mathbb{R}, \mathrm{id})$. The homeomorphism from Example 3.6 defines a new atlas $\mathscr{A}' := (\mathbb{R}, t \mapsto t^3)$.

1. Show that $\mathbb{R}$ equipped with the atlas $\mathscr{A}'$ is a 1-dimensional manifold.
2. Show that the maximal atlases generated by $\mathscr{A}$ and $\mathscr{A}'$ are different. Thus, we get two *distinct* manifolds $M = (\mathbb{R}, \mathscr{A})$ and $M' = (\mathbb{R}, \mathscr{A}')$.
   Hint: Recall that this mean that the union of these two atlases does not form an atlas.
3. Show that the map $f: M \to M'$ given by $f(x) = x^{1/3}$ is a diffeomorphism.

Reread Example 3.7 with this concrete example in mind.

Remark 3.1. In the introduction, we explained (without proof) the classification of 1-dimensional and 2-dimensional connected compact manifolds up to diffeomorphism. This classification coincides with their classification up to homeomorphism. This means, for example, that for any maximal atlas $\mathscr{A}'$ on S^2 which induces the same system of open subsets as the standard maximal atlas $\mathscr{A}$, there exists a homeomorphism $F: S^2 \to S^2$ taking $\mathscr{A}$ to $\mathscr{A}'$, in the sense that $(U, \varphi) \in \mathscr{A}$ if and only if

$(U', \varphi') \in \mathscr{A}'$, where $U' = F(U)$ and $\varphi' \circ F = \varphi$. In higher dimensions, it becomes much more complicated: .

It is quite possible for two manifolds to be homeomorphic but not diffeomorphic (unlike example 3.7). The first example of 'exotic' manifold structures was discovered by John Milnor in 1956, who found that the 7-sphere S^7 admits manifold structures that are not diffeomorphic to the standard manifold structure, but induce the standard topology. Kervaire and Milnor in 1963, proved that there are exactly 28 distinct manifold structures on S^7, and in fact classified all manifold structures on all spheres S^n with the exception of the case $n = 4$. For example, they showed that S^3, S^5, S^6 do not admit exotic (i.e., non-standard) manifold structures, while S^{15} has 16256 different manifold structures. For S^4 the existence of exotic manifold structures is an open problem; this is known as the *smooth Poincare conjecture*.

Around 1982, Michael Freedman (using results of Simon Donaldson) discovered the existence of exotic manifold structures on $\mathbb{R}^4$; later Clifford Taubes showed that there are uncountably many such. For $\mathbb{R}^n$ with $n \neq 4$, it is known that there are no exotic manifold structures on $\mathbb{R}^n$.

3.3 Examples of smooth maps

3.3.1 Products, diagonal maps

a) If M, N are manifolds, then the projection maps

$$\mathrm{pr}_M : M \times N \to M, \;\; \mathrm{pr}_N : M \times N \to N$$

are smooth. (This follows immediately by taking product charts $U_\alpha \times V_\beta$.)

b) The diagonal inclusion

$$\Delta_M : M \to M \times M$$

is smooth. (In a coordinate chart (U, φ) around p and the chart $(U \times U, \varphi \times \varphi)$ around (p, p), the map is the restriction to $\varphi(U) \subseteq \mathbb{R}^n$ of the diagonal inclusion $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$.)

c) Suppose $F : M \to N$ and $F' : M' \to N'$ are smooth maps. Then the direct product

$$F \times F' : M \times M' \to N \times N'$$

is smooth. This follows from the analogous statement for smooth maps on open subsets of Euclidean spaces.

3.3.2 The diffeomorphism $\mathbb{R}\mathrm{P}^1 \cong S^1$

We have stated before that $\mathbb{R}\mathrm{P}^1 \cong S^1$. To obtain an explicit diffeomorphism, we construct a bijection identifying the standard atlas for $\mathbb{R}\mathrm{P}^1$ with (essentially) the standard atlas for S^1. Recall that the atlas for $\mathbb{R}\mathrm{P}^1$ is given by

$$U_1 = \{(u:1)|\ u \in \mathbb{R}\},\ \ \varphi_1(u:1) = u,$$
$$U_0 = \{(1:u)|\ u \in \mathbb{R}\},\ \ \varphi_0(1:u) = u$$

with $\varphi_i(U_i) = \mathbb{R}$ and $\varphi_0(U_0 \cap U_1) = \varphi_1(U_0 \cap U_1) = \mathbb{R}\backslash\{0\}$, with the transition map $\varphi_1 \circ \varphi_0^{-1} :\ u \mapsto u^{-1}$. Similarly, the atlas for S^1 is

$$U_+ = \{(x,y) \in S^1|\ y \neq -1\}\ \ \varphi_+(x,y) = \frac{x}{1+y},$$
$$U_- = \{(x,y) \in S^1|\ y \neq +1\}\ \ \varphi_-(x,y) = \frac{x}{1-y}.$$

again with $\varphi_\pm(U_\pm) = \mathbb{R}$, $\varphi_\pm(U_+ \cap U_-) = \mathbb{R}\backslash\{0\}$, and transition map $u \mapsto u^{-1}$.

Hence, there is a well-defined diffeomorphism $F :\ \mathbb{R}\mathrm{P}^1 \to S^1$ which identifies the chart (U_-, φ_-) with (U_1, φ_1) and (U_+, φ_+) with (U_0, φ_0), in the sense that both

$$\varphi_- \circ F \circ \varphi_1^{-1} :\ \mathbb{R} \to \mathbb{R},\quad \varphi_+ \circ F \circ \varphi_0^{-1} :\ \mathbb{R} \to \mathbb{R}$$

are the identity $\mathrm{id}_\mathbb{R}$. Namely, the restriction of F to U_1 is $F_{U_1} = \varphi_-^{-1} \circ \varphi_1 :\ U_1 \to U_-$, the restriction to U_0 is $F|_{U_0} = \varphi_+^{-1} \circ \varphi_0 :\ U_0 \to U_+$. The inverse map $G = F^{-1} :\ S^1 \to \mathbb{R}\mathrm{P}^1$ is similarly given by $\varphi_0^{-1} \circ \varphi_+$ over U_+ and by $\varphi_1^{-1} \circ \varphi_-$ over U_-.

Exercise 31. Calculate the diffeomorphism $F : \mathbb{R}\mathrm{P}^1 \to S^1$ and its inverse $G : S^1 \to \mathbb{R}\mathrm{P}^1$ (give explicit formulas).

3.3.3 The diffeomorphism $\mathbb{C}\mathrm{P}^1 \cong S^2$

By a similar reasoning, we find $\mathbb{C}\mathrm{P}^1 \cong S^2$. For S^2 we use the atlas given by stereographic projection.

$$U_+ = \{(x,y,z) \in S^2|\ z \neq -1\}\ \ \varphi_+(x,y,z) = \frac{1}{1+z}\,(x,y),$$
$$U_- = \{(x,y,z) \in S^2|\ z \neq +1\}\ \ \varphi_-(x,y,z) = \frac{1}{1-z}\,(x,y).$$

The transition map is $u \mapsto \frac{u}{||u||^2}$, for $u = (u^1, u^2)$. Regarding u as a complex number $u = u^1 + iu^2$, the norm $||u||$ is just the absolute value of u, and the transition map becomes
$$u \mapsto \frac{u}{|u|^2} = \frac{1}{\overline{u}}.$$

Note that it is not quite the same as the transition map for the standard atlas of $\mathbb{C}\mathrm{P}^1$, which is given by $u \mapsto u^{-1}$. We obtain a unique diffeomorphism $F :\ \mathbb{C}\mathrm{P}^1 \to S^2$ such that $\varphi_+ \circ F \circ \varphi_0^{-1}$is the identity, while $\varphi_- \circ F \circ \varphi_1^{-1}$ is complex conjugation.

Exercise 32. Calculate the diffeomorphism $F : \mathbb{C}\mathrm{P}^1 \to S^2$ and its inverse $G : S^2 \to \mathbb{C}\mathrm{P}^1$.

3.3.4 Maps to and from projective space

We can now generalize Example 3.2. In Exercise 21 you verified that the quotient map

$$\pi : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}\mathrm{P}^n, \ x = (x^0, \ldots, x^n) \mapsto (x^0 : \ldots : x^n)$$

is smooth. Given a map $F : \mathbb{R}\mathrm{P}^n \to N$ to a manifold N, let $\widetilde{F} = F \circ \pi : \mathbb{R}^{n+1}\backslash\{0\} \to N$ be its composition with the projection map $\pi : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}\mathrm{P}^n$. That is,

$$\widetilde{F}(x^0, \ldots, x^n) = F(x^0 : \ldots : x^n).$$

Note that $\widetilde{F}(\lambda x^0 : \ldots : \lambda x^n) = \widetilde{F}(x^0, \ldots, x^n)$ for all non-zero λ; conversely, every map $\widetilde{F}$ with this property descends to a map F on projective space. We claim that the map F is smooth if and only the corresponding map $\widetilde{F}$ is smooth. One direction is clear: If F is smooth, then $\widetilde{F} = F \circ \pi$ is a composition of smooth maps. For the other direction, assuming that $\widetilde{F}$ is smooth, note that for the standard chart (U_j, φ_j), the maps

$$(F \circ \varphi_j^{-1})(u^1, \ldots, u^n) = \widetilde{F}(u^1, \ldots, u^i, 1, u^{i+1}, \ldots, u^n),$$

are smooth.

An analogous argument applies to the complex projective space $\mathbb{C}\mathrm{P}^n$, taking the x^i to be complex numbers z^i. That is, the quotient map $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}\mathrm{P}^n$ is smooth, and a map $F : \mathbb{C}\mathrm{P}^n \to N$ is smooth if and only if the corresponding map $\widetilde{F} : \mathbb{C}^{n+1}\backslash\{0\} \to N$ is smooth.

Exercise 33. The argument above demonstrates how to lift maps from projective spaces; show how we can use a similar technique to lift maps whose target is projective space.

As an application, we can see that the map

$$\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2, \ \ (z^0 : z^1) \mapsto ((z^0)^2 : (z^1)^2 : z^0 z^1)$$

is smooth, starting with the (obvious) fact that the lifted map

$$\mathbb{C}^2\backslash\{0\} \to \mathbb{C}^3\backslash\{0\}, \ \ (z^0, z^1) \mapsto ((z^0)^2, (z^1)^2, z^0 z^1)$$

is smooth.

3.3.5 The quotient map $S^{2n+1} \to \mathbb{C}\mathrm{P}^n$

As we explained above, the quotient map $q : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}\mathrm{P}^n$ is smooth. Since any class $[z] = (z^0 : \ldots : z^n)$ has a representative with $|z^0|^2 + \cdots + |z^n|^2 = 1$, and $|z^i|^2 =$

$(x^i)^2 + (y^i)^2$ for $z^i = x^i + \sqrt{-1}y^i$, we may also regard $\mathbb{C}\mathrm{P}^n$ as a set of equivalence classes in the unit sphere $S^{2n+1} \subseteq \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$. The resulting quotient map

$$\pi : S^{2n+1} \to \mathbb{C}\mathrm{P}^n$$

is again smooth, because it can be written as a composition of two smooth maps $\pi = q \circ \iota$ where $\iota : S^{2n+1} \to \mathbb{R}^{2n+2}\backslash\{0\} = \mathbb{C}^{n+1}\backslash\{0\}$ is the inclusion map.

For any $p \in \mathbb{C}\mathrm{P}^n$, the corresponding fiber $\pi^{-1}(p) \subseteq S^{2n+1}$ is diffeomorphic to a circle S^1 (which we may regard as complex numbers of absolute value 1). Indeed, given any point $(z^0, \ldots, z^n) \in \pi^{-1}(p)$ in the fiber, the other points are obtained as $(\lambda z^0, \ldots, \lambda z^n)$ where $|\lambda| = 1$.

In other words, we can think of

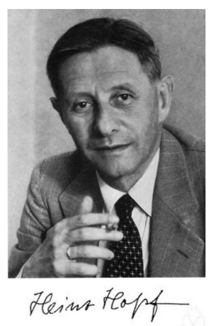
$$S^{2n+1} = \bigcup_{p \in \mathbb{C}\mathrm{P}^n} \pi^{-1}(p)$$

as a union of circles, parametrized by the points of $\mathbb{C}\mathrm{P}^n$. This is an example of what differential geometers call a *fiber bundle* or *fibration*. We won't give a formal definition here, but let us try to 'visualize' the fibration for the important case $n = 1$.

Identifying $\mathbb{C}\mathrm{P}^1 \cong S^2$ as above, the map π becomes a smooth map

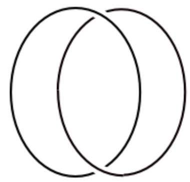
$$\pi : S^3 \to S^2$$

with fibers diffeomorphic to S^1. This map appears in many contexts; it is called the *Hopf fibration* (after *Heinz Hopf* (1894-1971)).

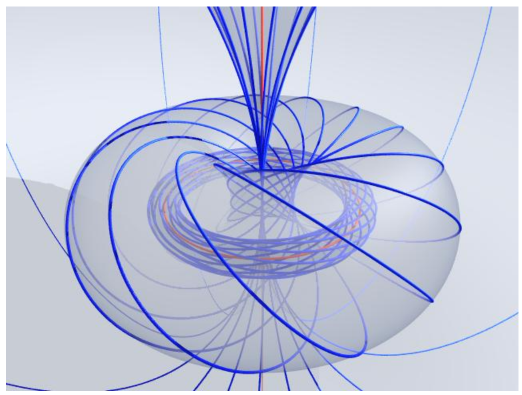


Let $S \in S^3$ be the 'south pole', and $N \in S^3$ the 'north pole'. We have that $S^3 - \{S\} \cong \mathbb{R}^3$ by stereographic projection. The set $\pi^{-1}(\pi(S)) - \{S\}$ projects to a straight line (think of it as a circle with 'infinite radius'). The fiber $\pi^{-1}(N)$ is a circle that goes around the straight line. If $Z \subseteq S^2$ is a circle at a given 'latitude', then $\pi^{-1}(Z)$ is is a 2-torus. For Z close to N this 2-torus is very thin, while for Z approaching the south pole S the radius goes to infinity. Each such 2-torus is itself a union of circles $\pi^{-1}(p),\ p \in Z$. Those circles are neither the usual 'vertical' or 'horizontal' circles of a 2-torus in $\mathbb{R}^3$, but instead are 'tilted'. In fact, each such circle is a 'perfect geometric circle' obtained as the intersection of its 2-torus with a carefully positioned affine 2-plane.

Moreover, any two of the circles $\pi^{-1}(p)$ are *linked*:



The full picture looks as follows:



A calculation shows that over the charts U_+, U_- (from stereographic projection), the Hopf fibration is just a product. That is, one has

$$\pi^{-1}(U_+) \cong U_+ \times S^1, \;\; \pi^{-1}(U_-) \cong U_- \times S^1.$$

In particular, the pre-image of the closed upper hemisphere is a *solid 2-torus* $D^2 \times S^1$ (with $D^2 = \{z \in \mathbb{C} | \; |z| \leq 1\}$ the unit disk), geometrically depicted as a 2-torus in $\mathbb{R}^3$ together with its interior.* We hence see that the S^3 may be obtained by gluing two solid 2-tori along their boundaries $S^1 \times S^1$.

* A solid torus is an example of a "manifold with boundary", a concept we haven't properly discussed yet.