3.5 Smooth maps of maximal rank

Let $F \in C^\infty(M,N)$ be a smooth map. Then the fibers (level sets)

$$F^{-1}(q) = \{x \in M |\ F(x) = q\}$$

for $q \in N$ need not be submanifolds, in general. Similarly, the image $F(M) \subseteq N$ need not be a submanifold – even if we allow self-intersections. (More precisely, there may be points p such that the image $F(U) \subseteq N$ of any open neighborhood U of p is never a submanifold.) Here are some counter-examples:

(a) The fibers $f^{-1}(c)$ of the map $f(x,y) = xy$ are hyperbolas for $c \neq 0$, but $f^{-1}(0)$ is the union of coordinate axes. What makes this possible is that the gradient of f is zero at the origin.

(b) As we mentioned earlier, the image of the smooth map

$$\gamma:\ \mathbb{R} \to \mathbb{R}^2,\ \ \gamma(t) = (t^2, t^3)$$

does not look smooth near $(0,0)$ (and replacing $\mathbb{R}$ by an open interval around 0 does not help)†. What makes this is possible is that the velocity $\dot\gamma(t)$ vanishes for $t = 0$: the curve described by γ 'comes to a halt' at $t = 0$, and then turns around.

In both cases, the problems arise at points where the map does not have maximal rank. After reviewing the notion of rank of a map from multivariable calculus we will generalize to manifolds.

3.5.1 The rank of a smooth map

The following discussion will involve some notions from multivariable calculus. Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets, and $F \in C^\infty(U,V)$ a smooth map.

Definition 3.5. *The* derivative *of F at $p \in U$ is the linear map*

$$D_pF : \mathbb{R}^m \to \mathbb{R}^n,\ \ v \mapsto \frac{d}{dt}\Big|_{t=0} F(p+tv).$$

The rank of F at *p is the rank of this linear map:*

$$\mathrm{rank}_p(F) = \mathrm{rank}(D_pF).$$

(Recall that the rank of a linear map is the dimension of its range.) Equivalently, D_pF is the $n \times m$ matrix of partial derivatives $(D_pF)^i_j = \frac{\partial F^i}{\partial x^j}\Big|_p$:

$$D_pF = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}\big|_p & \frac{\partial F^1}{\partial x^2}\big|_p & \cdots & \frac{\partial F^1}{\partial x^m}\big|_p \\ \frac{\partial F^2}{\partial x^1}\big|_p & \frac{\partial F^2}{\partial x^2}\big|_p & \cdots & \frac{\partial F^2}{\partial x^m}\big|_p \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F^n}{\partial x^1}\big|_p & \frac{\partial F^n}{\partial x^2}\big|_p & \cdots & \frac{\partial F^n}{\partial x^m}\big|_p \end{pmatrix}$$

† It is not a submanifold, although we have not proved it (yet).

and the rank of F at p is the rank of this matrix (i.e., the number of linearly independent rows, or equivalently the number of linearly independent columns). Note $\mathrm{rank}_p(F) \leq \min(m,n)$. By the chain rule for differentiation, the derivative of a composition of two smooth maps satisfies

$$D_p(F' \circ F) = D_{F(p)}(F') \circ D_p(F). \tag{3.10}$$

In particular, if F' is a diffeomorphism then $\mathrm{rank}_p(F' \circ F) = \mathrm{rank}_p(F)$, and if F is a diffeomorphism then $\mathrm{rank}_p(F' \circ F) = \mathrm{rank}_{F(p)}(F')$.

Definition 3.6. *Let $F \in C^\infty(M,N)$ be a smooth map between manifolds, and $p \in M$. The* rank *of F at $p \in M$ is defined as*

$$\mathrm{rank}_p(F) = \mathrm{rank}_{\varphi(p)}(\psi \circ F \circ \varphi^{-1})$$

for any two coordinate charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U) \subseteq V$.

By (3.10), this is well-defined: if we use different charts (U',φ') and (V',ψ'), then the rank of

$$\psi' \circ F \circ (\varphi')^{-1} = (\psi' \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ (\varphi')^{-1})$$

at $\varphi'(p)$ equals that of $\psi \circ F \circ \varphi^{-1}$ at $\varphi(p)$, since the two maps are related by diffeomorphisms.

The following discussion will focus on maps of *maximal rank*. We have that

$$\mathrm{rank}_p(F) \leq \min(\dim M,\ \dim N)$$

for all $p \in M$; the map F is said to have *maximal rank* at p if $\mathrm{rank}_p(F) = \min(\dim M,\ \dim N)$. A point $p \in M$ is called a *critical point* for F if $\mathrm{rank}_p(F) < \min(\dim M,\ \dim N)$.

Exercise 40.

1. Consider the lemniscate of Gerono: $f : \mathbb{R} \to \mathbb{R}^2$

$$\theta \mapsto (\cos\theta, \sin\theta\cos\theta).$$

   Find its critical points. What is $\mathrm{rank}_p(f)$ for p not a critical point?
2. Consider the map $g : \mathbb{R}^3 \to \mathbb{R}^4$

$$(x,y,z) \mapsto (yz, xy, xz, x^2 + 2y^2 + 3z^2).$$

   Find its critical points, and for each critical point p compute $\mathrm{rank}_p(F)$.

3.5.2 Local diffeomorphisms

In this section we will consider the case $\dim M = \dim N$. Our ‘workhorse theorem’ from multivariable calculus is going to be the following fact.

Theorem 3.1 (Inverse Function Theorem for $\mathbb{R}^m$). *Let $F \in C^\infty(U,V)$ be a smooth map between open subsets of $\mathbb{R}^m$, and suppose that the derivative D_pF at $p \in U$ is invertible. Then there exists an open neighborhood $U_1 \subseteq U$ of p such that F restricts to a diffeomorphism $U_1 \to F(U_1)$.*

Remark 3.5. The theorem tells us that for a smooth bijection, a sufficient condition for smoothness of the inverse map is that the differential (i.e., the *first* derivative) is invertible everywhere. It is good to see, in just one dimension, how this is possible. Given an invertible smooth function $y = f(x)$, with inverse $x = g(y)$, and using $\frac{\mathrm{d}}{\mathrm{d}y} = \frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}x}$, we have

$$\begin{aligned} g'(y) &= \frac{1}{f'(x)}, \\ g''(y) &= = \frac{-f''(x)}{f'(x)^3}, \\ g'''(y) &= = \frac{-f'''(x)}{f'(x)^4} + 3\frac{f''(x)^2}{f'(x)^5}, \end{aligned}$$

and so on; only powers of $f'(x)$ appear in the denominator.

Theorem 3.2 (Inverse function theorem for manifolds). *Let $F \in C^\infty(M,N)$ be a smooth map between manifolds of the same dimension $m = n$. If $p \in M$ is such that $\mathrm{rank}_p(F) = m$, then there exists an open neighborhood $U \subseteq M$ of p such that F restricts to a diffeomorphism $U \to F(U)$.*

Proof. Choose charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U) \subseteq V$. The map

$$\widetilde{F} = \psi \circ F \circ \varphi^{-1} : \ \widetilde{U} := \varphi(U) \to \widetilde{V} := \psi(V)$$

has rank m at $\varphi(p)$. Hence, by the inverse function theorem for $\mathbb{R}^m$, after replacing $\widetilde{U}$ with a smaller open neighborhood of $\varphi(p)$ (equivalently, replacing U with a smaller open neighborhood of p) the map $\widetilde{F}$ becomes a diffeomorphism from $\widetilde{U}$ onto $\widetilde{F}(\widetilde{U}) = \psi(F(U))$. It then follows that

$$F = \psi^{-1} \circ \widetilde{F} \circ \varphi : \ U \to V$$

is a diffeomorphism $U \to F(U)$. □

A smooth map $F \in C^\infty(M,N)$ is called a *local diffeomorphism* if $\dim M = \dim N$, and F has maximal rank everywhere. By the theorem, this is equivalent to the condition that every point p has an open neighborhood U such that F restricts to a diffeomorphism $U \to F(U)$. It depends on the map in question which of these two conditions is easier to verify.

Exercise 41. Show that the map $\mathbb{R} \to S^1, \ t \mapsto (\cos(2\pi t), \ \sin(2\pi t))$ is a local diffeomorphism.

Example 3.13. The quotient map $\pi: S^n \to \mathbb{R}\mathrm{P}^n$ is a local diffeomorphism. Indeed, one can see (using suitable coordinates) that π restricts to diffeomorphisms from each $U_j^{\pm} = \{x \in S^n |\ \pm x^j > 0\}$ to the standard chart U_j.

Example 3.14. Let M be a manifold with a countable open cover $\{U_\alpha\}$, and let

$$Q = \bigsqcup_\alpha U_\alpha$$

be the *disjoint* union. Then the map $\pi: Q \to M$, given on $U_\alpha \subseteq Q$ by the inclusion into M, is a local diffeomorphism. Since π is surjective, it determines an equivalence relation on Q, with π as the quotient map and $M = Q/\sim$.

We leave it as an exercise to show that if the U_α's are the domains of coordinate charts, then Q is diffeomorphic to an open subset of $\mathbb{R}^m$. This then shows that any manifold is realized as a quotient of an open subset of $\mathbb{R}^m$, in such a way that the quotient map is a local diffeomorphism.

Exercise 42. Continuing with the notation of the last example above, show that if the U_α's are the domain of coordinate charts, then Q is diffeomorphic to an open subset of $\mathbb{R}^m$.

3.5.3 Level sets, submersions

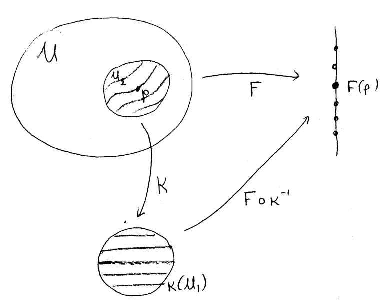
The inverse function theorem is closely related to the *implicit function theorem*, and one may be obtained as a consequence of the other. (We have chosen to take the inverse function theorem as our starting point.)

Proposition 3.8. *Suppose $F \in C^\infty(U,V)$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, and suppose $p \in U$ is such that the derivative D_pF is surjective. Then there exists an open neighborhood $U_1 \subseteq U$ of p and a diffeomorphism $\kappa: U_1 \to \kappa(U_1) \subseteq \mathbb{R}^m$ such that*

$$(F \circ \kappa^{-1})(u^1, \ldots, u^m) = (u^{m-n+1}, \ldots, u^m)$$

for all $u = (u^1, \ldots, u^m) \in \kappa(U_1)$.

Thus, in suitable coordinates F is given by a projection onto the last n coordinates.



Although it belongs to multivariable calculus, let us recall how to get this result from the inverse function theorem.

Proof. The idea is to extend F to a map between open subsets of $\mathbb{R}^m$, and then apply the inverse function theorem.

By assumption, the derivative D_pF has rank equal to n. Hence it has n linearly independent columns. By re-indexing the coordinates of $\mathbb{R}^m$ (this permutation is itself a change of coordinates) we may assume that these are the last n columns. That is, writing

$$D_pF = \Big(C, D\Big)$$

where C is the $n \times (m-n)$-matrix formed by the first $m-n$ columns and D the $n \times n$-matrix formed by the last n columns, the square matrix D is invertible. Write elements $x \in \mathbb{R}^m$ in the form $x = (x', x'')$ where x' are the first $m-n$ coordinates and x'' the last n coordinates. Let

$$G\colon\ U \to \mathbb{R}^m,\ x = (x', x'') \mapsto (x', F(x)).$$

Then the derivative D_pG has block form

$$D_pG = \begin{pmatrix} I_{m-n} & 0 \\ C & D \end{pmatrix},$$

(where I_{m-n} is the square $(m-n) \times (m-n)$ matrix), and is therefore invertible. Hence, by the inverse function theorem there exists a smaller open neighborhood U_1 of p such that G restricts to a diffeomorphism $\kappa\colon\ U_1 \to \kappa(U_1) \subseteq \mathbb{R}^m$. We have,

$$G \circ \kappa^{-1}(u', u'') = (u', u'')$$

for all $(u', u'') \in \kappa(U_1)$. Since F is just G followed by projection to the x'' component, we conclude

$$F \circ \kappa^{-1}(u', u'') = u''.$$

□

Again, this result has a version for manifolds:

Theorem 3.3. *Let $F \in C^\infty(M, N)$ be a smooth map between manifolds of dimensions $m \geq n$, and suppose $p \in M$ is such that* $\mathrm{rank}_p(F) = n$. *Then there exist coordinate charts (U, φ) around p and (V, ψ) around $F(p)$, with $F(U) \subseteq V$, such that*

$$(\psi \circ F \circ \varphi^{-1})(u', u'') = u''$$

for all $u = (u', u'') \in \varphi(U)$. In particular, for all $q \in V$ the intersection

$$F^{-1}(q) \cap U$$

is a submanifold of dimension $m - n$.

Proof. Start with coordinate charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U)\subseteq V$. Apply Proposition 3.8 to the map $\widetilde{F}=\psi\circ F\circ\varphi^{-1}:\ \varphi(U)\to\psi(V)$, to define a smaller neighborhood $\varphi(U_1)\subseteq\varphi(U)$ and change of coordinates κ so that $\widetilde{F}\circ\kappa^{-1}(u',u'')=u''$. After renaming $(U_1,\kappa\circ\varphi|_{U_1})$ as (U,φ) we have the desired charts for F. The last part of the theorem follows since (U,φ) becomes a submanifold chart for $F^{-1}(q)\cap U$ (after shifting φ by $\psi(q)\in\mathbb{R}^n$). □

Definition 3.7. *Let $F\in C^\infty(M,N)$. A point $q\in N$ is called a* regular value *of $F\in C^\infty(M,N)$ if for all $x\in F^{-1}(q)$, one has* $\mathrm{rank}_x(F)=\dim N$. *It is called a* critical value *(sometimes also* singular value*) if it is not a regular value.*

Note that regular values are only possible if $\dim N\le\dim M$. Note also that all points of N that are not in the image of the map F are considered regular values. We may restate Theorem 3.3 as follows:

Theorem 3.4 (Regular Value Theorem). *For any regular value $q\in N$ of a smooth map $F\in C^\infty(M,N)$, the level set $S=F^{-1}(q)$ is a submanifold of dimension*

$$\dim S=\dim M-\dim N.$$

Example 3.15. The n-sphere S^n may be defined as the level set $F^{-1}(1)$ of the function $F\in C^\infty(\mathbb{R}^{n+1},\mathbb{R})$ given by

$$F(x^0,\dots,x^n)=(x^0)^2+\cdots+(x^n)^2.$$

The derivative of F is the $1\times(n+1)$-matrix of partial derivatives, that is, the gradient ∇F:

$$D_pF=(2x^0,\dots,2x^n).$$

For $x\neq 0$ this has maximal rank. Note that any nonzero real number q is a regular value since $0\notin F^{-1}(q)$. Hence all the level sets $F^{-1}(q)$ for $q\neq 0$ are submanifolds.

Exercise 43. Let $0<r<R$. Show that

$$F(x,y,z)=(\sqrt{x^2+y^2}-R)^2+z^2$$

has r^2 as a regular value. (The corresponding level set the 2-torus.)

Example 3.16. The *orthogonal group* $\mathrm{O}(n)$ is the group of matrices $A\in\mathrm{Mat}_{\mathbb{R}}(n)$ satisfying $A^\top=A^{-1}$. We claim that $\mathrm{O}(n)$ is a submanifold of $\mathrm{Mat}_{\mathbb{R}}(n)$. To see this, consider the map

$$F:\ \mathrm{Mat}_{\mathbb{R}}(n)\to\mathrm{Sym}_{\mathbb{R}}(n),\ \ A\mapsto A^\top A,$$

where $\mathrm{Sym}_{\mathbb{R}}(n)\subseteq\mathrm{Mat}_{\mathbb{R}}(n)$ denotes the subspace of symmetric matrices. We want to show that the identity matrix I is a regular value of F. We compute the differential $D_AF:\ \mathrm{Mat}_{\mathbb{R}}(n)\to\mathrm{Sym}_{\mathbb{R}}(n)$ using the definition‡

‡ Note that it would have been confusing to work with the description of D_AF as a matrix of partial derivatives.

$$\begin{aligned}(D_A F)(X) &= \frac{d}{dt}\Big|_{t=0} F(A+tX) \\ &= \frac{d}{dt}\Big|_{t=0} ((A^\top + tX^\top)(A+tX)) \\ &= A^\top X + X^\top A.\end{aligned}$$

To see that this is surjective, for $A \in F^{-1}(I)$, we need to show that for any $Y \in \mathrm{Sym}_{\mathbb{R}}(n)$ there exists a solution of

$$A^\top X + X^\top A = Y.$$

Using $A^\top A = F(A) = I$ we see that $X = \frac{1}{2}AY$ is a solution. We conclude that I is a regular value, and hence that $\mathrm{O}(n) = F^{-1}(I)$ is a submanifold. Its dimension is

$$\dim \mathrm{O}(n) = \dim \mathrm{Mat}_{\mathbb{R}}(n) - \dim \mathrm{Sym}_{\mathbb{R}}(n) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1).$$

Note that it was important here to regard F as a map to $\mathrm{Sym}_{\mathbb{R}}(n)$; for F viewed as a map to $\mathrm{Mat}_{\mathbb{R}}(n)$ the identity would *not* be a regular value.

Exercise 44. In using the regular value theorem we have implicitly assumed that $\mathrm{Sym}_{\mathbb{R}}(n)$ is a manifold. Prove that this is indeed the case.

Definition 3.8. *A smooth map $F \in C^\infty(M,N)$ is a* submersion *if* $\mathrm{rank}_p(F) = \dim N$ *for all $p \in M$.*

Thus, for a submersion *all* level sets $F^{-1}(q)$ are submanifolds.

Example 3.17. Local diffeomorphisms are submersions; here the level sets $F^{-1}(q)$ are discrete points, i.e. 0-dimensional manifolds.

Example 3.18. Recall that $\mathbb{C}\mathrm{P}^n$ can be regarded as a quotient of S^{2n+1}. Using charts, one can check that the quotient map $\pi:\ S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is a submersion. Hence its fibers $\pi^{-1}(q)$ are 1-dimensional submanifolds. Indeed, as discussed before these fibers are circles. As a special case, the Hopf fibration $S^3 \to S^2$ is a submersion.

Remark 3.6. (For those who are familiar with quaternions.) Let $\mathbb{H} = \mathbb{C}^2 = \mathbb{R}^4$ be the quaternion numbers. The unit quaternions are a 3-sphere S^3. Generalizing the definition of $\mathbb{R}\mathrm{P}^n$ and $C\mathrm{P}^n$, there are also quaternion projective spaces, $\mathbb{H}\mathrm{P}^n$. These are quotients of the unit sphere inside $\mathbb{H}^{n+1}$, hence one obtains submersions

$$S^{4n+3} \to \mathbb{H}\mathrm{P}^n;$$

the fibers of this submersion are diffeomorphic to S^3. For $n = 1$, one can show that $\mathbb{H}\mathrm{P}^1 = S^4$, hence one obtains a submersion

$$\pi:\ S^7 \to S^4$$

with fibers diffeomorphic to S^3.

3.5.4 Example: The Steiner surface

In this section, we will give more lengthy examples, investigating the smoothness of level sets.

Example 3.19 (Steiner's surface). Let $S \subseteq \mathbb{R}^3$ be the solution set of

$$y^2z^2 + x^2z^2 + x^2y^2 = xyz.$$

in $\mathbb{R}^3$. Is this a smooth surface in $\mathbb{R}^3$? (We use *surface* as another term for *2-dimensional manifold*; by a *surface in M* we mean a 2-dimensional submanifold.) Actually, we can easily see that it's *not*. If we take one of x, y, z equal to 0, then the equation holds if and only if one of the other two coordinates is 0. Hence, the intersection of S with the set where $xyz = 0$ (the union of the coordinate hyperplanes) is the union of the three coordinate axes.

Hence, let us rephrase the question: Letting $U \subseteq \mathbb{R}^3$ be the subset where $xyz \neq 0$, is $S \cap U$ is surface? To investigate the problem, consider the function

$$f(x, y, z) = y^2z^2 + x^2z^2 + x^2y^2 - xyz.$$

The differential (which in this case is the same as the gradient) is the 1×3-matrix

$$D_{(x,y,z)}f = \begin{pmatrix} 2x(y^2 + z^2) - yz & 2y(z^2 + x^2) - zx & 2z(x^2 + y^2) - xy \end{pmatrix}$$

This vanishes if and only if all three entries are zero. Vanishing of the first entry gives, after dividing by $2xy^2z^2$, the condition

$$\frac{1}{z^2} + \frac{1}{y^2} = \frac{1}{2xyz};$$

we get similar conditions after cyclic permutation of x, y, z. Thus we have

$$\frac{1}{z^2} + \frac{1}{y^2} = \frac{1}{x^2} + \frac{1}{z^2} = \frac{1}{y^2} + \frac{1}{x^2} = \frac{1}{2xyz},$$

with a unique solution $x = y = z = \frac{1}{4}$. Thus, $D_{(x,y,z)}f$ has maximal rank (i.e., it is nonzero) except at this point. But this point doesn't lie on S. We conclude that $S \cap U$ is a submanifold. How does it look like? It turns out that there is a nice answer. First, let's divide the equation for $S \cap U$ by xyz. The equation takes on the form

$$xyz(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}) = 1. \tag{3.11}$$

The solution set of (**??**) is cantoned in the set of all (x, y, z) such that $xyz > 0$. On this subset, we introduce new variables

$$\alpha = \frac{\sqrt{xyz}}{x}, \ \ \beta = \frac{\sqrt{xyz}}{y}, \ \ \gamma = \frac{\sqrt{xyz}}{z};$$

the old variables x,y,z are recovered as

$$x = \beta\gamma, \; y = \alpha\gamma, \; z = \alpha\beta.$$

In terms of α, β, γ, Equation (3.11) becomes the equation $\alpha^2 + \beta^2 + \gamma^2 = 1$. Actually, it is even better to consider the corresponding points

$$(\alpha : \beta : \gamma) = (\frac{1}{x} : \frac{1}{y} : \frac{1}{z}) \in \mathbb{R}\mathrm{P}^2,$$

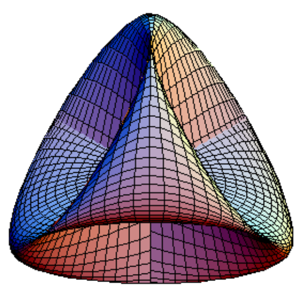
because we could take either square root of xyz (changing the sign of all α, β, γ doesn't affect x,y,z). We conclude that the map $U \to \mathbb{R}\mathrm{P}^2$, $(x,y,z) \mapsto (\frac{1}{x} : \frac{1}{y} : \frac{1}{z})$ restricts to a diffeomorphism from $S \cap U$ onto

$$\mathbb{R}\mathrm{P}^2 \backslash \{(\alpha : \beta : \gamma) | \; \alpha\beta\gamma = 0\}.$$

The image of the map

$$\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3, \; (\alpha : \beta : \gamma) \mapsto \frac{1}{|\alpha|^2 + |\beta|^2 + |\gamma|^2}(\beta\gamma, \alpha\beta, \alpha\gamma).$$

is called *Steiner's surface*, even though it is not a submanifold (not even an *immersed* submanifold). Here is a picture:



Note that the subset of $\mathbb{R}\mathrm{P}^2$ defined by $\alpha\beta\gamma = 0$ is a union of three $\mathbb{R}\mathrm{P}^1 \cong S^1$, each of which maps into a coordinate axis (but not the entire coordinate axis). For example, the circle defined by $\alpha = 0$ maps to the set of all $(0,0,z)$ with $-\frac{1}{2} \le z \le \frac{1}{2}$. In any case, S is the Steiner surface together with the three coordinate axes.

Example 3.20. Let $S \subseteq \mathbb{R}^4$ be the solution set of

$$y^2x^2 + x^2z^2 + x^2y^2 = xyz, \quad y^2x^2 + 2x^2z^2 + 3x^2y^2 = xyzw.$$

Again, this cannot quite be a surface because it contains the coordinate axes for x,y,z. Closer investigation shows that S is the union of the three coordinate axes, together with the image of an injective map

$$\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^4, \; (\alpha : \beta : \gamma) \mapsto \frac{1}{\alpha^2 + \beta^2 + \gamma^2}(\beta\gamma, \alpha\beta, \alpha\gamma, \alpha^2 + 2\beta^2 + 3\gamma^2).$$

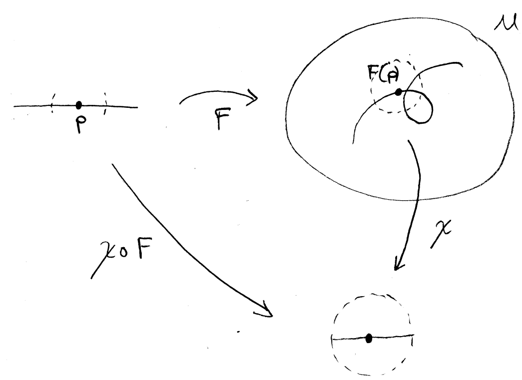
It turns out (see Section 4.2.4 below) that the latter is a submanifold, which realizes $\mathbb{R}\mathrm{P}^2$ as a surface in $\mathbb{R}^4$.

3.5.5 Immersions

We next consider maps $F: M \to N$ of maximal rank between manifolds of dimensions $m \le n$. Once again, such a map can be put into a 'normal form': By choosing suitable coordinates it becomes linear.

Proposition 3.9. *Suppose $F \in C^\infty(U,V)$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, and suppose $p \in U$ is such that the derivative D_pF is injective. Then there exist smaller neighborhoods $U_1 \subseteq U$ of p and $V_1 \subseteq V$ of $F(p)$, with $F(U_1) \subseteq V_1$, and a diffeomorphism $\chi: V_1 \to \chi(V_1)$, such that*

$$(\chi \circ F)(u) = (u,0) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$$



Proof. Since D_pF is injective, it has m linearly independent rows. By re-indexing the rows (which amounts to a change of coordinates on V), we may assume that these are the first m rows.

That is, writing

$$D_pF = \begin{pmatrix} A \\ C \end{pmatrix}$$

where A is the $m \times m$-matrix formed by the first m rows and C is the $(n-m) \times m$-matrix formed by the last $n-m$ rows, the square matrix A is invertible. Consider the map

$$H: U \times \mathbb{R}^{n-m} \to \mathbb{R}^n,\ (x,y) \mapsto F(x) + (0,y)$$

Then

$$D_{(p,0)}H = \begin{pmatrix} A & 0 \\ C & I_{n-m} \end{pmatrix}$$

is invertible. Hence, by the inverse function theorem for $\mathbb{R}^n$, H is a diffeomorphism from some neighborhood of $(p,0)$ in $U \times \mathbb{R}^{n-m}$ onto some neighborhood V_1 of $H(p,0) = F(p)$, which we may take to be contained in V. Let

$$\chi:\ V_1 \to \chi(V_1) \subseteq U \times \mathbb{R}^{n-m}$$

be the inverse; thus

$$(\chi \circ H)(x,y) = (x,y)$$

for all $(x,y) \in \chi(V_1)$. Replace U with the smaller open neighborhood

$$U_1 = F^{-1}(V_1) \cap U$$

of p. Then $F(U_1) \subseteq V_1$, and

$$(\chi \circ F)(u) = (\chi \circ H)(u,0) = (u,0)$$

for all $u \in U_1$. □

The manifolds version reads as follows:

Theorem 3.5. *Let $F \in C^\infty(M,N)$ be a smooth map between manifolds of dimensions $m \leq n$, and $p \in M$ a point with $\mathrm{rank}_p(F) = m$. Then there are coordinate charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U) \subseteq V$ and*

$$(\psi \circ F \circ \varphi^{-1})(u) = (u,0).$$

In particular, $F(U) \subseteq N$ is a submanifold of dimension m.

Proof. Once again, this is proved by introducing charts around p, $F(p)$ to reduce to a map between open subsets of $\mathbb{R}^m$, $\mathbb{R}^n$, and then use the multivariable version of the result to obtain a change of coordinates, putting the map into normal form. □

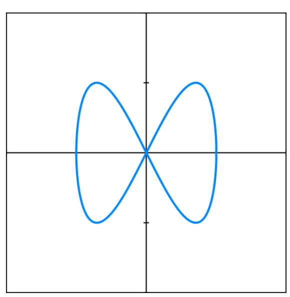
Definition 3.9. *A smooth map $F:\ M \to N$ is an* immersion *if* $\mathrm{rank}_p(F) = \dim M$ *for all $p \in M$.*

Example 3.21. Let $J \subseteq \mathbb{R}$ be an open interval. A smooth map $\gamma:\ J \to M$ is also called a *smooth curve*. We see that the image of γ is an immersed submanifold, provided that $\mathrm{rank}_p(\gamma) = 1$ for all $p \in M$. In local coordinates (U,φ), this means that $\frac{d}{dt}(\varphi \circ \gamma)(t) \neq 0$ for all t with $\gamma(t) \in U$. For example, the curve $\gamma(t) = (t^2, t^3)$ fails to have this property at $t = 0$.

Example 3.22 (Figure eight). The map

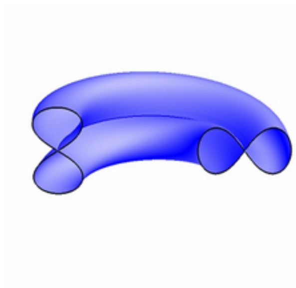
$$\gamma:\ \mathbb{R} \to \mathbb{R}^2,\ \ t \mapsto \big(\sin(t), \sin(2t)\big)$$

is an immersion; the image is a figure eight.



(Indeed, for all $t \in \mathbb{R}$ we have $D_t\gamma \equiv \dot{\gamma}(t) \neq 0$.)

Example 3.23 (Immersion of the Klein bottle). The Klein bottle admits a 'figure eight' immersion into $\mathbb{R}^3$, obtained by taking the figure eight in the $x-z$-plane, moving in the x-direction by $R > 1$, and then rotating about the z-axis while at the same time rotating the figure eight, so that after a full turn $\varphi \mapsto \varphi + 2\pi$ the figure eight has performed a half turn.



We can regard this procedure as a composition of the following maps:

$$\begin{aligned} F_1 :\ & (t,\varphi) \mapsto (\sin(t), \sin(2t), \varphi) = (u,v,\varphi),\\ F_2 :\ & (u,v,\varphi) \mapsto \Big(u\cos(\frac{\varphi}{2}) + v\sin(\frac{\varphi}{2}),\ v\cos(\frac{\varphi}{2}) - u\sin(\frac{\varphi}{2}), \varphi\Big) = (a,b,\varphi)\\ F_3 :\ & \big(a,b,\varphi\big) \mapsto ((a+R)\cos\varphi,\ (a+R)\sin\varphi,\ b\big) = (x,y,z). \end{aligned}$$

Here F_1 is the figure eight in the $u-v$-plane (with φ just a bystander). F_2 rotates the $u-v$-plane as it moves in the direction of φ, by an angle of $\varphi/2$; thus $\varphi = 2\pi$ corresponds to a half-turn. The map F_3 takes this family of rotating u,v-planes, and wraps it around the circle in the $x-y$-plane of radius R, with φ now playing the role of the angular coordinate.

The resulting map $F = F_3 \circ F_2 \circ F_1 :\ \mathbb{R}^2 \to \mathbb{R}^3$ is given by $F(t,\varphi) = (x,y,z)$, where with

$$\begin{aligned} x &= \Big(R + \cos(\frac{\varphi}{2})\sin(t) + \sin(\frac{\varphi}{2})\sin(2t)\Big)\cos\varphi,\\ y &= \Big(R + \cos(\frac{\varphi}{2})\sin(t) + \sin(\frac{\varphi}{2})\sin(2t)\Big)\sin\varphi,\\ z &= \cos(\frac{\varphi}{2})\sin(2t) - \sin(\frac{\varphi}{2})\sin(t) \end{aligned}$$

is an immersion. To verify that this is an immersion, it would be cumbersome to work out the Jacobian matrix directly. It is much easier to use that F is obtained as a composition $F = F_3 \circ F_2 \circ F_1$ of the three maps considered above, where F_1 is an immersion, F_2 is a diffeomorphism, and F_3 is a local diffeomorphism from the open subset where $|a| < R$ onto its image.

Since the right hand side of the equation for F does not change under the transformations

$$(t,\varphi) \mapsto (t+2\pi,\varphi), \quad (t,\varphi) \mapsto (-t,\varphi+2\pi),$$

this descends to an immersion of the Klein bottle. It is straightforward to check that this immersion of the Klein bottle is injective, except over the 'central circle' corresponding to $t=0$, where it is 2-to-1.

Note that under the above construction, any point of the figure eight creates a circle after two'full turns', $\varphi \mapsto \varphi+4\pi$. The complement of the circle generated by the point $t=\pi/2$ consists of two subsets of the Klein bottle, generated by the parts of the figure eight defined by $-\pi/2 < t < \pi/2$, and by $\pi/2 < t < 3\pi/2$. Each of these is a 'curled-up' immersion of an open Möbius strip. (Remember, it is possible to remove a circle from the Klein bottle to create two Möbius strips!) The point $t=0$ also creates a circle; its complement is the subset of the Klein bottle generated by $0<t<2\pi$. (Remember, it is possible to remove a circle from a Klein bottle to create one Möbius strip.) We can also remove one copy of the figure eight itself; then the 'rotation' no longer matters and the complement is an open cylinder. (Remember, it is possible to remove a circle from a Klein bottle to create a cylinder.)

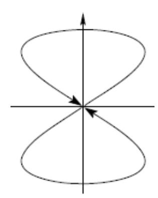
Example 3.24. Let M be a manifold, and $S \subseteq M$ a k-dimensional submanifold. Then the inclusion map $\iota: \ S \to M, \ \ x \mapsto x$ is an immersion. Indeed, if (V,ψ) is a submanifold chart for S, with $p \in U = V \cap S, \ \varphi = \psi|_{V\cap S}$ then

$$(\psi \circ F \circ \varphi^{-1})(u) = (u,0),$$

which shows that

$$\mathrm{rank}_p(F) = \mathrm{rank}_{\varphi(p)}(\psi \circ F \circ \varphi^{-1}) = k.$$

By an *embedding*, we will mean an immersion given as the inclusion map for a submanifold. Not every injective immersion is an embedding; the following picture gives a counter-example:

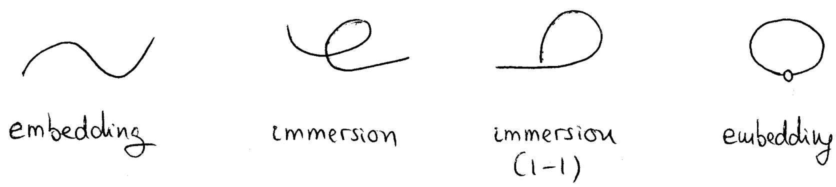


In practice, showing that an injective smooth map is an immersion tends to be easier than proving that its image is a submanifold. Fortunately, for compact manifolds we have the following fact:

Theorem 3.6. *If M is a* compact *manifold, then every injective immersion $F: \ M \to N$ is an embedding as a submanifold $S = F(M)$.*

Proof. Let $p \in M$ be given. By Theorem 3.5, we can find charts (U,φ) around p and (V,ψ) around $F(p)$, with $F(U) \subseteq V$, such that $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$ is in normal form: i.e., $\widetilde{F}(u) = (u,0)$. We would like to take (V,ψ) as a submanifold chart for $S = F(M)$, but this may not work yet since $F(M) \cap V = S \cap V$ may be strictly larger than $F(U) \cap V$. Note however that $A := M \backslash U$ is compact, hence its image $F(A)$ is compact, and therefore closed (here we are using that N is Haudorff). Since F is injective, we have that $p \notin F(A)$. Replace V with the smaller open neighborhood $V_1 = V \backslash (V \cap F(A))$. Then $(V_1, \psi|_{V_1})$ is the desired submanifold chart.

Remark 3.7. Some authors refer to injective immersions $\iota :\ S \to M$ as 'submanifolds' (thus, a submanifold is taken to be a map rather than a subset). To clarify, 'our' submanifolds are sometimes called 'embedded submanifolds' or 'regular submanifolds'.



Example 3.25. Let A, B, C be distinct real numbers. We will leave it as a homework problem to verify that the map

$$F :\ \mathbb{R}\mathrm{P}^2 \to \mathbb{R}^4,\ \ (\alpha : \beta : \gamma) \mapsto (\beta\gamma, \alpha\gamma, \alpha\beta,\ A\alpha^2 + B\beta^2 + C\gamma^2),$$

where we use representatives (α, β, γ) such that $\alpha^2 + \beta^2 + \gamma^2 = 1$, is an injective immersion. Hence, by Theorem 3.6, it is an embedding of $\mathbb{R}\mathrm{P}^2$ as a submanifold of $\mathbb{R}^4$.

To summarize the outcome from the last few sections: If $F \in C^\infty(M, N)$ has maximal rank near $p \in M$, then one can always choose local coordinates around p and around $F(p)$ such that the coordinate expression of F becomes a linear map of maximal rank. (This simple statement contains the inverse and implicit function theorems from multivariable calculus are special cases.)

Remark 3.8. This generalizes further to maps of *constant rank*. In fact, if $\mathrm{rank}_p(F)$ is independent of p on some open subset U, then for all $p \in U$ one can choose coordinates in which F becomes linear.