5.1 Vector Fields as Derivations

A vector field on a manifold may be regarded as a family of tangent vectors $X_p \in T_pM$ for $p \in M$, depending smoothly on the base points $p \in M$. One way of making precise what is meant by 'depending smoothly' is the following.

Definition 5.1 (Vector fields – first definition). A collection of tangent vectors X_p , $p \in M$ defines a vector field $X \in \mathfrak{X}(M)$ if and only if for all functions $f \in C^{\infty}(M)$ the function $p \mapsto X_p(f)$ is smooth. The space of all vector fields on M is denoted $\mathfrak{X}(M)$.

We hence obtain a linear map $X : C^{\infty}(M) \to C^{\infty}(M)$ such that

$$X(f)(p) = X(f)|_{p} = X_{p}(f).$$
(5.1)

Since each X_p satisfy the product rule (at p), it follows that X itself satisfies a product rule. We can use this as an alternative definition:

Definition 5.2 (Vector fields – second definition). A vector field on M is a linear map

$$X: C^{\infty}(M) \to C^{\infty}(M)$$

satisfying the product rule,

$$X(fg) = X(f)g + fX(g)$$
(5.2)

for $f,g \in C^{\infty}(M)$.

Remark 5.1. The condition (5.2) says that X is a *derivation* of the algebra $C^{\infty}(M)$ of smooth functions. More generally, a derivation of an algebra A is a linear map $D: A \to A$ such that

$$D(a_1a_2) = D(a_1) a_2 + a_1 D(a_2).$$

We can also express the smoothness of the tangent vectors X_p in terms of coordinate charts (U, φ) . Recall that for any $p \in U$, and all $f \in C^{\infty}(M)$, the tangent vector X_p is expressed as

$$X_p(f) = \sum_{i=1}^m a^i \frac{\partial}{\partial u^i} \Big|_{u=\varphi(p)} (f \circ \varphi^{-1}).$$

The vector $a = (a^1, ..., a^m) \in \mathbb{R}^m$ represents X_p in the chart; i.e., $(T_p \varphi)(X_p) = a$ under the identification $T_{\varphi(p)}\varphi(U) = \mathbb{R}^m$. As p varies in U, the vector a becomes a function of $p \in U$, or equivalently of $u = \varphi(p)$.

Proposition 5.1. The collection of tangent vectors X_p , $p \in M$ define a vector field if and only if for all charts (U, φ) , the functions $a^i : \varphi(U) \to \mathbb{R}$ defined by

$$X_{\varphi^{-1}(u)}(f) = \sum_{i=1}^{m} a^{i}(u) \frac{\partial}{\partial u^{i}} (f \circ \varphi^{-1}),$$

are smooth.

Proof. If the a^i are smooth functions, then for every $f \in C^{\infty}(M)$ the function $X(f) \circ \varphi^{-1}$: $\varphi(U) \to \mathbb{R}$ is smooth, and hence $X(f)|_U$ is smooth. Since this is true for all charts, it follows that X(f) is smooth. Conversely, if X is a vector field, and $p \in M$ some point in a coordinate chart (U, φ) , and $i \in \{1, \ldots, m\}$ a given index, choose $f \in C^{\infty}(M)$ such that $f(\varphi^{-1}(u)) = u^i$. Then $X(f) \circ \varphi^{-1} = a^i(u)$, which shows that the a^i are smooth.

Exercise 59. In the proof, we used that for any coordinate chart (U, φ) around p, one can choose $f \in C^{\infty}(M)$ such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ coincides with u^j near $\varphi(p)$. Write out the details in the construction of such a function f, using a choice of 'bump function'.

In particular, we see that vector fields on open subsets $U \subseteq \mathbb{R}^m$ are of the form

$$X = \sum_{i} a^{i} \frac{\partial}{\partial x^{i}}$$

where $a^i \in C^{\infty}(U)$. Under a diffeomorphism $F : U \to V, x \mapsto y = F(x)$, the coordinate vector fields transform with the Jacobian

$$TF\left(\frac{\partial}{\partial x^{i}}\right) = \sum_{j} \frac{\partial F^{j}}{\partial x^{i}}\Big|_{x=F^{-1}(y)} \frac{\partial}{\partial y^{j}}$$

Informally, this 'change of coordinates' is often written

$$\frac{\partial}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}.$$

Here one thinks of the x^i and y^j as coordinates on the same set, and doesn't worry about writing coordinate maps, and one uses the (somewhat sloppy, but convenient) notation y = y(x) instead of y = F(x).

Exercise 60. Consider \mathbb{R}^3 with coordinates *x*, *y*, *z*. Introduce new coordinates *u*, *v*, *w* by setting

 $x = e^{u}v$; $y = e^{v}$; $z = uv^{2}w$

valid on the region where $x \ge y > 1$.

(a) Express u, v, w in terms of x, y, z.

(b) Express the coordinate vector fields $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial v}$, $\frac{\partial}{\partial w}$ as a combination of the coordinate vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ with coefficients functions of *x*, *y*, *z*.

5.2 Vector Fields as Sections of the Tangent Bundle

The 'best' way of describing the smoothness of $p \mapsto X_p$ is that it is literally a smooth map into the tangent bundle.

Definition 5.3 (Vector fields – third definition). A vector field on M is a smooth map $X \in C^{\infty}(M, TM)$ such that $\pi \circ X$ is the identity.

Remark 5.2. In the condition $\pi \circ X$, we take $\pi : TM \to M$ the canonical projection map that sends each (p, v) to its base-point p. The condition simply states that at each point $p \in M$ we have that X(p) is of the form (p, v) for $v \in T_pM$; i.e. it expresses the idea that X_p should "live in" T_pM .

It is common practice to use the same symbol *X* both as a linear map from smooth functions to smooth functions, or as a map into the tangent bundle. Thus,

 $X: M \to TM, \quad X: C^{\infty}(M) \to C^{\infty}(M)$

coexist. But if it gets too confusing, one uses a symbol

$$L_X: C^{\infty}(M) \to C^{\infty}(M)$$

for the interpretation as a derivation; here the L stands for 'Lie derivative' (named after Sophus Lie). Both viewpoints are useful and important, and both have their advantages and disadvantages. For instance, from the tangent-bundle viewepoint it is immediate that vector fields on M restrict to open subsets $U \subseteq M$; this map

$$\mathfrak{X}(M) \to \mathfrak{X}(U), \ X \mapsto X|_U$$

may seem a little awkward from the derivations viewpoint since $C^{\infty}(U)$ is not a subspace of $C^{\infty}(M)$. (There is a restriction map $C^{\infty}(M) \to C^{\infty}(U)$, but no natural map in the other direction.) On the other hand, the derivations viewpoint gives the *Lie bracket operation* discussed below, which seems unexpected from the tangent-bundle viewpoint.

Exercise 61. The tangent-bundle viewpoint provides another easy way of specifying vector fields.

- (a) Show that the mapping $p \mapsto (p,0)$ is a vector field, and write it in coordinates. This is called the *zero section*. Show that it is an embedding of M into TM.
- (b) Consider \mathbb{R}^n and identify $T_p \mathbb{R}^n \cong \mathbb{R}^n$. Show that the mapping $p \mapsto (p, p)$ is a vector field, write it in coordinates, and draw a picture of it in the case n = 2.

5.3 Lie brackets

Let *M* be a manifold. Given vector fields $X, Y : C^{\infty}(M) \to C^{\infty}(M)$, the composition $X \circ Y$ is not a vector field: For example, if $X = Y = \frac{\partial}{\partial x}$ as vector fields on \mathbb{R} , then $X \circ Y = \frac{\partial^2}{\partial x^2}$ is a second order derivative, which is not a vector field (it does not satisfy the Leibnitz rule). However, the commutator turns out to be a vector field:

Theorem 5.1. For any two vector fields $X, Y \in \mathfrak{X}(M)$ (regarded as derivations), the commutator

$$[X,Y] := X \circ Y - Y \circ X : C^{\infty}(M) \to C^{\infty}(M)$$

is again a vector field.

Exercise 62. Prove the theorem by using the second definition of vector fields.

Remark 5.3. A similar calculation applies to derivations of algebras in general: The commutator of two derivations is again a derivations.

Definition 5.4. The vector field

$$[X,Y] := X \circ Y - Y \circ X$$

is called the Lie bracket *of* $X, Y \in \mathfrak{X}(M)$ *.*

It is instructive to see how this works in local coordinates. For open subsets $U \subseteq \mathbb{R}^m$, if

$$X = \sum_{i=1}^{m} a^{i} \frac{\partial}{\partial x^{i}}, \quad Y = \sum_{i=1}^{m} b^{i} \frac{\partial}{\partial x^{i}}$$

with coefficient functions $a^i, b^i \in C^{\infty}(U)$, the composition $X \circ Y$ is a second order differential operators on functions $f \in C^{\infty}(U)$:

$$X \circ Y = \sum_{i=1}^{m} \sum_{j=1}^{m} a^{j} \frac{\partial b^{i}}{\partial x^{j}} \frac{\partial}{\partial x^{i}} + \sum_{i=1}^{m} \sum_{j=1}^{m} a^{i} b^{j} \frac{\partial^{2}}{\partial x^{i} \partial x^{j}}$$

Subtracting a similar expression for $Y \circ X$, the terms involving second derivatives cancel, and we obtain

$$[X,Y] = \sum_{i=1}^{m} \sum_{j=1}^{m} \left(a^{j} \frac{\partial b^{i}}{\partial x^{j}} - b^{j} \frac{\partial a^{i}}{\partial x^{j}} \right) \frac{\partial}{\partial x^{i}}$$

(This calculation applies to general manifolds, by taking local coordinates.) The significance of the Lie bracket will become clear later. At this stage, let us give some examples.

Exercise 63. Compute the Lie bracket [X, Y] for the following vector fields in \mathbb{R}^2 :

- (a) $X_1 = \frac{\partial}{\partial x}, Y_1 = \frac{\partial}{\partial y}.$
- (b) $X_2 = \frac{\partial}{\partial x}, Y_2 = (1 + x^2) \frac{\partial}{\partial y}$ (c) Notice that X_2 and Y_2 are linearly independent everywhere. Is it possible to introduce coordinates $(u, v) = \varphi(x, y)$, such that in the new coordinates, these vector fields are the coordinate vector fields $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$?

Note: When calculating Lie brackets $X \circ Y - Y \circ X$ of vector fields X, Y in local coordinates, it is not necessary to work out the second order derivatives - we know in advance that these are going to cancel out!

Exercise 64. Consider the following two vector fields on \mathbb{R}^2 , on the open subset where xy > 0, $X = \frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad Y = 2\sqrt{xy}\frac{\partial}{\partial x}$ (a) Compute their Lie bracket [X, Y]. (b) Define a change of coordinates u, v by $x = uv^2$; y = u. Express the coordinate vector fields $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ in terms of the coordinate vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ (with coefficients functions of x, y).

Example 5.1. Consider the same problem for the vector fields

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \ Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

This time, we may verify that [X, Y] = 0. Introduce polar coordinates,

$$x = r\cos\theta, \ y = r\sin\theta.$$

(this is a well-defined coordinate chart for r > 0 and $-\pi < \theta < \pi$). We have *

^{*} In the following, we are using somewhat sloppy notation. Given $(\theta, r) = \varphi(x, y)$, we should more properly write $\varphi_* X, \varphi_* Y$ for the vector fields in the new coordinates.

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \frac{1}{r}Y$$

and

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} = X$$

Hence $X = \frac{\partial}{\partial \theta}$, $Y = r \frac{\partial}{\partial r}$. To get this into the desired form, we make another change of coordinates $\rho = f(r)$ in such a way that *Y* becomes $\frac{\partial}{\partial \rho}$. Since

$$\frac{\partial}{\partial r} = \frac{\partial \rho}{\partial r} \frac{\partial}{\partial \rho} = f'(r) \frac{\partial}{\partial \rho}$$

we want $f'(r) = \frac{1}{r}$, thus $f(r) = \ln(r)$. So, $r = e^{\rho}$. Hence, the desired change of coordinates is

$$x = e^{\rho} \cos \theta, \ y = e^{\rho} \sin \theta.$$

Definition 5.5. Let $S \subseteq M$ be a submanifold. A vector field $X \in \mathfrak{X}(M)$ is called tangent to *S* if for all $p \in S$, the tangent vector X_p lies in $T_pS \subseteq T_pM$. (Thus X restricts to a vector field $X|_S \in \mathfrak{X}(S)$.)

Exercise 65.

(a) Show that the three vector fields

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

on \mathbb{R}^3 are tangent to the 2-sphere S^2 .

(b) Compute the bracket [*X*,*Y*], [*Y*,*Z*], [*Z*,*X*] and show that the resulting vector fields are again tangent to the 2-sphere.

Proposition 5.2. *If two vector fields* $X, Y \in \mathfrak{X}(M)$ *are tangent to a submanifold* $S \subseteq M$, *then their Lie bracket is again tangent to S.*

Proposition 5.2 can be proved by using the coordinate expressions of X, Y in submanifold charts. But we will postpone the proof for now since there is a much shorter, coordinate-independent proof (see the next section).

Exercise 66. Consider the vector fields on \mathbb{R}^3 ,

$$X = \frac{\partial}{\partial x}, \ Y = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}.$$

Show that there cannot exist a surface $S \subseteq \mathbb{R}^3$ such that both *X* and *Y* are tangent to *S*.

Hint: compute the bracket and show that the three fields form a basis.

Exercise 67. Verify the Jacobi identity holds for any three vector fields $X, Y, Z \in \mathfrak{X}(M)$:

[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.

5.4 Related vector fields

Definition 5.6. Let $F \in C^{\infty}(M,N)$ be a smooth map. Vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called *F*-related, written as

$$X \sim_F Y$$

if

$$T_p F(X_p) = Y_{F(p)}$$

for all $p \in M$.

Exercise 68.

- (a) Suppose $F: M \to N$ is a diffeomorphism. Show that $X \sim_F Y$ if and only if $Y = F_*X$. (In particular, if N = M, then an equation $X \sim_F X$ means that X is *invariant* under *F*.)
- (b) Suppose $S \subseteq M$ is an embedded submanifold, and $\iota : S \hookrightarrow M$ the inclusion map. Let $X \in \mathfrak{X}(S)$ and $Y \in \mathfrak{X}(M)$. Show that

 $X \sim_i Y$

if and only if *Y* is tangent to *S*, with *X* as its restriction. (In particular, $0 \sim_i Y$ if and only if *Y* vanishes along the submanifold *S*.)

(c) Suppose $F: M \to N$ is a submersion, and $X \in \mathfrak{X}(M)$. Show that $X \sim_F 0$ if and only if X is tangent to the fibers of F.

Example 5.2. Let $\pi : S^n \to \mathbb{R}P^n$ be the quotient map. Then $X \sim_{\pi} Y$ if and only if the vector field X is invariant under the transformation $F : S^n \to S^n$, $x \mapsto -x$ (that is, $TF \circ X = X \circ F$, and with Y the induced vector field on the quotient.

The F-relation of vector fields also has a simple interpretation in terms of the 'differential operator' picture.

Proposition 5.3. One has $X \sim_F Y$ if and only if for all $g \in C^{\infty}(N)$,

$$X(g \circ F) = Y(g) \circ F.$$

In terms of the pull-back notation, with $F^*g = g \circ F$ for $g \in C^{\infty}(N)$, this means $X \circ F^* = F^* \circ Y$:



Proof. The condition $X(g \circ F) = Y(g) \circ F$ says that

$$(T_p F(X_p))(g) = Y_{F(p)}(g)$$

for all $p \in M$. \Box

The key fact concerning related vector fields is the following.

Theorem 5.2. Let $F \in C^{\infty}(M,N)$ For vector fields $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(M)$, we have

$$X_1 \sim_F Y_1, \quad X_2 \sim_F Y_2 \Rightarrow \quad [X_1, X_2] \sim_F [Y_1, Y_2].$$

Exercise 69. Use the differential operator picture to prove Theorem 5.2.

Exercise 70. Prove Proposition 5.2: If two vector fields Y_1, Y_2 are tangent to a submanifold $S \subseteq M$ then their Lie bracket $[Y_1, Y_2]$ is again tangent to *S*, and the Lie bracket of their restriction is the restriction of the Lie brackets.

Exercise 71. Show that in the description of vector fields as sections of the tangent bundle, two vector fields $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(M)$ are *F*-related if and only if the following diagram commutes:

$$TM \xrightarrow{TF} TN$$

$$x \uparrow \qquad \uparrow Y$$

$$M \xrightarrow{F} N$$

5.5 Flows of vector fields

For any curve $\gamma: J \to M$, with $J \subseteq \mathbb{R}$ an open interval, and any $t \in J$, the velocity vector

$$\dot{\gamma}(t) \equiv rac{d\gamma}{dt} \in T_{\gamma(t)}M$$

is defined as the tangent vector, given in terms of its action on functions as

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$$(\dot{\gamma}(t))(f) = \frac{d}{dt}f(\gamma(t)).$$

(The dot signifies a *t*-derivative.) The curve representing this tangent vector for a given *t*, in the sense of our earlier definition, is the shifted curve $\tau \mapsto \gamma(t + \tau)$. Equivalently, one may think of the velocity vector as the image of $\frac{\partial}{\partial t}|_t \in T_t J \cong \mathbb{R}$ under the tangent map $T_t \gamma$:

$$\dot{\gamma}(t) = (T_t \gamma)(\frac{\partial}{\partial t}|_t).$$

Definition 5.7. Suppose $X \in \mathfrak{X}(M)$ is a vector field on a manifold M. A smooth curve $\gamma \in C^{\infty}(J,M)$, where $J \subseteq \mathbb{R}$ is an open interval, is called a solution curve to X if

$$\dot{\gamma}(t) = X_{\gamma(t)} \tag{5.3}$$

for all $t \in J$.

Geometrically, Equation (5.3) means that at any given time *t*, the value of *X* at $\gamma(t)$ agrees with the velocity vector to γ at *t*.



Equivalently, in terms of related vector fields,

$$\frac{\partial}{\partial t}\sim_{\gamma} X.$$

Consider first the case that $M = U \subseteq \mathbb{R}^m$. Here curves $\gamma(t)$ are of the form

$$\gamma(t) = x(t) = (x^1(t), \dots, x^m(t)),$$

hence

$$\dot{\gamma}(t)(f) = \frac{d}{dt}f(x(t)) = \sum_{i=1}^{m} \frac{\mathrm{d}x^{i}}{\mathrm{d}t} \frac{\partial f}{\partial x^{i}}(x(t))$$

That is

$$\dot{\gamma}(t) = \sum_{i=1}^{m} \frac{\mathrm{d}x^{i}}{\mathrm{d}t} \frac{\partial}{\partial x^{i}}\Big|_{x(t)}.$$

On the other hand, the vector field has the form $X = \sum_{i=1}^{m} a^{i}(x) \frac{\partial}{\partial x^{i}}$. Hence (5.3) becomes the system of first order ordinary differential equations,

$$\frac{\mathrm{d}x^{i}}{\mathrm{d}t} = a^{i}(x(t)), \quad i = 1, \dots, m.$$
 (5.4)

Example 5.3. The solution curves of the coordinate vector field $\frac{\partial}{\partial x^j}$ are of the form

$$x^{i}(t) = x_{0}^{i}, i \neq j, x^{j}(t) = x_{0}^{j} + t.$$

More generally, if $a = (a^1, ..., a^m)$ is a constant function of x (so that $X = \sum a^i \frac{\partial}{\partial x^i}$ is the *constant vector field*, the solution curves are affine lines,

$$x(t) = x_0 + ta$$

Exercise 72. Consider the vector field on \mathbb{R}^2 ,

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Find the solution curve for any given $(x_0, y_0) \in \mathbb{R}^2$. Draw a picture of the vector field.

Exercise 73. Consider the following vector field on \mathbb{R}^m ,

$$X = \sum_{i=1}^{m} x^{i} \frac{\partial}{\partial x^{i}}.$$

Find solution curves for any given $(x_0^1, \ldots, x_0^m) \in \mathbb{R}^m$.

One of the main results from the theory of ODE's says that for any given initial condition $x(0) = x_0$, a solution to the system (5.4) exists and is (essentially) unique:

Theorem 5.3 (Existence and uniqueness theorem for ODE's). Let $U \subseteq \mathbb{R}^m$ be an open subset, and $a \in C^{\infty}(U, \mathbb{R}^m)$. For any given $x_0 \in U$, there is an open interval $J_{x_0} \subseteq \mathbb{R}$ around 0, and a solution $x : J_{x_0} \to U$ of the ODE

$$\frac{dx^i}{dt} = a^i(x(t)), \ i = 1, \dots, m$$

with initial condition $x(0) = x_0$, and which is maximal in the sense that any other solution to this initial value problem is obtained by restriction to some subinterval of J_{x_0} .

Thus, J_{x_0} is the maximal open interval on which the solution is defined. The solution depends smoothly on initial conditions, in the following sense. For any given x_0 , let $\Phi(t,x_0)$ be the solution x(t) of the initial value problem with initial condition x_0 , that is,

$$\Phi(0,x_0) = x_0, \ \frac{d}{dt}\Phi(t,x_0) = a(\Phi(t,x_0)).$$

Theorem 5.4 (Dependence on initial conditions for ODE's). For $a \in C^{\infty}(U, \mathbb{R}^m)$ as above, the set

$$\mathscr{J} = \{(t,x) \in \mathbb{R} \times U \mid t \in J_x\}.$$

is an open neighborhood of $\{0\} \times U$ in $\mathbb{R} \times U$, and the map

$$\Phi: \mathscr{J} \to U, \ (t,x) \mapsto \Phi(t,x)$$

is smooth.

In general, the interval J_{x_0} may be strictly smaller than \mathbb{R} , because a solution might escape to infinity in finite time.

Exercise 74. For each of the following ODEs: find solution curves with initial condition x_0 ; find J_{x_0} , \mathscr{J} , and $\Phi(t,x)$. 1. $\dot{x} = 1$ on $U = (0,1) \subseteq \mathbb{R}$. 2. $\dot{x} = x^2$ on $U = \mathbb{R}$. 3. $\dot{x} = 1 + x^2$ on $U = \mathbb{R}$.

For a general vector field $X \in \mathfrak{X}(M)$ on manifolds, Equation (5.3) becomes (5.4) after introduction of local coordinates. In detail: Let (U, φ) be a coordinate chart. In the chart, *X* becomes the vector field

$$\varphi_*(X) = \sum_{j=1}^m a^j(u) \frac{\partial}{\partial u^j}$$

and $\varphi(\gamma(t)) = u(t)$ with

$$\dot{u}^i = a^i(u(t)).$$

If $a = (a^1, ..., a^m)$: $\varphi(U) \to \mathbb{R}^m$ corresponds to X in a local chart (U, φ) , then any solution curve $x : J \to \varphi(U)$ for a defines a solution curve $\gamma(t) = \varphi^{-1}(x(t))$ for X. The existence and uniqueness theorem for ODE's extends to manifolds, as follows:

Theorem 5.5 (Solutions of vector fields on manifolds). Let $X \in \mathfrak{X}(M)$ be a vector field on a manifold M. For any given $p \in M$, there is an open interval $J_p \subseteq \mathbb{R}$ around 0, and a solution $\gamma : J_p \to M$ of the initial value problem

$$\dot{\gamma}(t) = X_{\gamma(t)}, \quad \gamma(0) = p, \tag{5.5}$$

which is maximal in the sense that any other solution of the initial value problem is obtained by restriction to a subinterval. The set

$$\mathscr{J} = \{(t, p) \in \mathbb{R} \times M | t \in J_p\}$$

is an open neighborhood of $\{0\} \times M$, and the map

$$\Phi: \mathscr{J} \to M, (t,p) \mapsto \Phi(t,p)$$

such that $\gamma(t) = \Phi(t, p)$ solves the initial value problem (5.5), is smooth.

Proof. Existence and uniqueness of solutions for small times *t* follows from the existence and uniqueness theorem for ODE's, by considering the vector field in local charts. To prove uniqueness even for large times *t*, let $\gamma : J \rightarrow M$ be a maximal solution of (5.5) (i.e., a solution that cannot be extended to a larger open interval), and let $\gamma_1 : J_1 \rightarrow M$ be another solution of the same initial value problem, but with $\gamma_1(t) \neq \gamma(t)$ for some $t \in J$, t > 0. (There is a similar discussion if the solution is different for some t < 0). Then we can define

$$b = \inf\{t \in J \mid t > 0, \ \gamma_1(t) \neq \gamma(t)\}.$$

By the uniqueness for small t, we have b > 0. We will get a contradiction in both of the following cases:

Case 1: $\gamma_1(b) = \gamma(b) =: q$. Then both $\lambda_1(s) = \gamma_1(b+s)$ and $\lambda(s) = \gamma(b+s)$ are solutions to the initial value problem

$$\lambda(0) = q, \quad \lambda(s) = X_{\lambda(s)};$$

hence they have to agree for small |s|, and consequently $\gamma_1(t), \gamma(t)$ have to agree for *t* close to *b*. This contradicts the definition of *b*.

Case 2: $\gamma_1(b) \neq \gamma(b)$. Using the Hausdorff property of M, we can choose disjoint open neighborhoods U of $\gamma(b)$ and U_1 of $\gamma(b_1)$. For $t = b - \varepsilon$ with $\varepsilon > 0$ sufficiently small, $\gamma(t) \in U$ while $\gamma_1(t) \in U_1$. But this is impossible since $\gamma(t) = \gamma_1(t)$ for $0 \le t < b$.

The result for ODE's about the smooth dependence on initial conditions shows, by taking local coordinate charts, that \mathscr{J} contains an open neighborhood of $\{0\} \times M$, on which Φ is given by a smooth map. The fact that \mathscr{J} itself is open, and the map Φ is smooth everywhere, follows by the 'flow property' to be discussed below. (We will omit the details of this part of the proof.) \Box

Note that the uniqueness part uses the Hausdorff property in the definition of manifolds. Indeed, the uniqueness part may fail for non-Hausdorff manifolds.

Example 5.4. A counter-example is the non-Hausdorff manifold

$$Y = (\mathbb{R} \times \{1\}) \cup (\mathbb{R} \times \{-1\}) / \sim,$$

where \sim glues two copies of the real line along the strictly negative real axis. Let U_{\pm} denote the charts obtained as images of $\mathbb{R} \times \{\pm 1\}$. Let X be the vector field on

Y, given by $\frac{\partial}{\partial x}$ in both charts. It is well-defined, since the transition map is just the identity map. Then $\gamma_+(t) = \pi(t, 1)$ and $\gamma_-(t) = \pi(t, -1)$ are both solution curves, and they agree for negative *t* but not for positive *t*.

Given a vector field X, the map $\Phi : \mathscr{J} \to M$ is called the *flow* of X. For any given p, the curve $\gamma(t) = \Phi(t, p)$ is a solution curve. But one can also fix t and consider the time-t flow,

$$\Phi_t(p) \equiv \Phi(t,p).$$

It is a smooth map Φ_t : $U_t \to M$, defined on the open subset

$$U_t = \{ p \in M | (t, p) \in \mathscr{J} \}.$$

Note that $\Phi_0 = id_M$.

Intuitively, $\Phi_t(p)$ is obtained from the initial point $p \in M$ by flowing for time *t* along the vector field *X*. One expects that first flowing for time *t*, and then flowing for time *s*, should be the same as flowing for time t + s. Indeed one has the following *flow property*.

Theorem 5.6 (Flow property). Let $X \in \mathfrak{X}(M)$, with flow $\Phi : \mathcal{J} \to M$. Let $(t_2, p) \in \mathcal{J}$, and $t_1 \in \mathbb{R}$. Then

$$(t_1, \Phi_{t_2}(p)) \in \mathscr{J} \Leftrightarrow (t_1 + t_2, p) \in \mathscr{J},$$

and one has

$$\Phi_{t_1}(\Phi_{t_2}(p)) = \Phi_{t_1+t_2}(p).$$

Proof. Given $t_2 \in J_p$, we consider both sides as functions of $t_1 = t$. Write $q = \Phi_{t_2}(p)$. We claim that both

$$t \mapsto \Phi_t(\Phi_{t_2}(p)), \quad t \mapsto \Phi_{t+t_2}(p)$$

are maximal solution curves of *X*, for the same initial condition *q*. This is clear for the first curve, and follows for the second curve by the calculation, for $f \in C^{\infty}(M)$,

$$\frac{d}{dt}f(\Phi_{t+t_2}(p)) = \frac{d}{ds}\Big|_{s=t+t_2} \Phi_s(p) = X_{\Phi_s(p)}(f)\Big|_{s=t+t_2} = X_{\Phi_{t+t_2}(p)}(f).$$

Hence, the two curves must coincide. The domain of definition of $t \mapsto \Phi_{t+t_2}(p)$ is the interval \mathscr{J}_p , shifted by t_2 . Hence, $t_1 \in J_{\Phi(t_2,p)}$ if and only if $t_1 + t_2 \in J_p$. \Box

We see in particular that for any *t*, the map $\Phi_t : U_t \to M$ is a diffeomorphism onto its image $\Phi_t(U_t) = U_{-t}$, with inverse Φ_{-t} .

Example 5.5. Let us illustrate the flow property for various vector fields on \mathbb{R} . The flow property is evident for $\frac{\partial}{\partial x}$ with flow $\Phi_t(x) = x + t$, as well as for $x \frac{\partial}{\partial x}$, with flow $\Phi_t(x) = e^t x$. The vector field $x^2 \frac{\partial}{\partial x}$ has flow $\Phi_t(x) = x/(1-tx)$, defined for 1-tx < 1. We can explicitly verify the flow property:

$$\Phi_{t_1}(\Phi_{t_2}(x)) = \frac{\Phi_{t_2}(x)}{1 - t_1 \Phi_{t_2}(x)} = \frac{\frac{x}{1 - t_2 x}}{1 - t_1 \frac{x}{1 - t_2 x}} = \frac{x}{1 - (t_1 + t_2)x} = \Phi_{t_1 + t_2}(x).$$

Let *X* be a vector field, and $\mathcal{J} = \mathcal{J}^X$ be the domain of definition for the flow $\Phi = \Phi^X$.

Definition 5.8. A vector field $X \in \mathfrak{X}(M)$ is called complete if $\mathscr{J}^X = R \times M$.

Thus X is complete if and only if all solution curves exist for all time.

Example 5.6. The vector field $x \frac{\partial}{\partial x}$ on $M = \mathbb{R}$ is complete, but $x^2 \frac{\partial}{\partial x}$ is incomplete.

A vector field may fail to be complete if a solution curve escapes to infinity in finite time. This suggests that a vector fields *X* that vanishes outside a compact set must be complete, because the solution curves are 'trapped' and cannot escape to infinity.

Proposition 5.4. If $X \in \mathfrak{X}(M)$ is a vector field that has compact support, in the sense that $X|_{M-A} = 0$ for some compact subset A, then X is complete. In particular, every vector field on a compact manifold is complete.

Proof. By the uniqueness theorem for solution curves γ , and since X vanishes outside A, if $\gamma(t_0) \in M - A$ for some t_0 , then $\gamma(t) = \gamma(t_0)$ for all t. Hence, if a solution curve $\gamma : J \to M$ has $\gamma(0) \in A$, then $\gamma(t) \in A$ for all t. Let $U_{\varepsilon} \subseteq M$ be the set of all p such that the solution curve γ with initial condition $\gamma(0) = p$ exists for $|t| < \varepsilon$ (that is, $(-\varepsilon, \varepsilon) \subseteq J_p$). By smooth dependence on initial conditions, U_{ε} is open. The collection of all U_{ε} with $\varepsilon > 0$ covers A, since every solution curve exists for sufficiently small time. Since A is compact, there exists a finite subcover $U_{\varepsilon_1}, \ldots, U_{\varepsilon_k}$. Let ε be the smallest of $\varepsilon_1, \ldots, \varepsilon_k$. Then $U_{\varepsilon_i} \subseteq U_{\varepsilon}$, for all i, and hence $A \subseteq U_{\varepsilon}$. Hence, for any $p \in A$ we have $(-\varepsilon, \varepsilon) \subseteq J_p$, that is any solution curve $\gamma(t)$ starting in A exists for times $|t| < \varepsilon$. But $\gamma(-\varepsilon/2), \gamma(\varepsilon/2) \in A$, hence the solution curve starting at those points again exist for times $< \varepsilon$. This shows $(-3\varepsilon/2, 3\varepsilon/2) \subseteq J_p$. Continuing in this way, we find that $(-\varepsilon - N\varepsilon/2, \varepsilon + N\varepsilon/2) \subseteq J_p$ for all N, thus $J_p = \mathbb{R}$ for all $p \in A$. For points $p \in M - A$, it is clear anyhow that $J_p = \mathbb{R}$, since the solution curves are constant. \Box

Theorem 5.7. If X is a complete vector field, the flow Φ_t defines a 1-parameter group of diffeomorphisms. That is, each Φ_t is a diffeomorphism and

$$\Phi_0 = \mathrm{id}_M, \ \Phi_{t_1} \circ \Phi_{t_2} = \Phi_{t_1+t_2}$$

Conversely, if Φ_t is a 1-parameter group of diffeomorphisms such that the map $(t, p) \mapsto \Phi_t(p)$ is smooth, the equation

$$X_p(f) = \frac{d}{dt}\Big|_{t=0} f(\Phi_t(p))$$

defines a complete vector field X on M, with flow Φ_t .

Proof. It remains to show the second statement. Given Φ_t , the linear map

$$C^{\infty}(M) \to C^{\infty}(M), \quad f \mapsto \frac{d}{dt}\Big|_{t=0} f(\Phi_t(p))$$

satisfies the product rule, hence it is a vector field *X*. Given $p \in M$ the curve $\gamma(t) = \Phi_t(p)$ is an integral curve of *X* since

$$\frac{d}{dt}\Phi_t(p) = \frac{d}{ds}\Big|_{s=0}\Phi_{t+s}(p) = \frac{d}{ds}\Big|_{s=0}\Phi_s(\Phi_t(p)) = X_{\Phi_t(p)}.$$

Remark 5.4. In terms of pull-backs, the relation between the vector field and its flow reads as

$$\frac{d}{dt}\Phi_t^*(f) = \Phi_t^* \frac{d}{ds}\Big|_{s=0} \Phi_s^*(f) = \Phi_t^* X(f).$$

This identity

$$\frac{d}{dt}\Phi_t^* = \Phi_t^* \circ X$$

as linear maps $C^{\infty}(M) \to C^{\infty}(M)$ may be viewed as the definition of the flow.

Example 5.7. Given $A \in Mat_{\mathbb{R}}(m)$ let

$$\Phi_t: \mathbb{R}^m \to \mathbb{R}^m, \ x \mapsto e^{tA}x = \Big(\sum_{j=0}^{\infty} \frac{t^j}{j!} A^j\Big)x$$

(using the exponential map of matrices). Since $e^{(t_1+t_2)A} = e^{t_1A}e^{t_2A}$, and since $(t,x) \mapsto e^{tA}x$ is a smooth map, Φ_t defines a flow. What is the corresponding vector field *X*? For any function $f \in C^{\infty}(\mathbb{R}^m)$ we calculate,

$$X(f)(x) = \frac{d}{dt}\Big|_{t=0} f(e^{tA}x)$$
$$= \sum_{j} \frac{\partial f}{\partial x^{j}} (Ax)^{j}$$
$$= \sum_{ij} A_{i}^{j} x^{i} \frac{\partial f}{\partial x^{j}}$$

showing that

$$X = \sum_{ij} A_i{}^j x^i \frac{\partial}{\partial x^j}.$$

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As a special case, taking *A* to be the identity matrix, we recover the Euler vector field $X = \sum_{i} x^{i} \frac{\partial}{\partial x^{i}}$, and its flow $\Phi_{t}(x) = e^{t}x$.

$$Ax = \sum_{ij} A_i{}^j x^i e_j$$

from which we read off $(Ax)^j = \sum_i A_i^{\ j} x^i$.

[†] Here we wrote the matrix entries for the *i*-th row and *j*-th column as A_i^{j} rather than A_{ij} . That is, one standard basis vectors $e_i \in \mathbb{R}^m$ (written as column vectors), we have $A(e_i) = \sum_j A_i^{j} e_j$, hence for $x = \sum x^i e_i$ we get

Example 5.8. Let *X* be a complete vector field, with flow Φ_t . For each $t \in \mathbb{R}$, the tangent map $T\Phi_t : TM \to TM$ has the flow property,

$$T\Phi_{t_1} \circ T\Phi_{t_2} = T(\Phi_{t_1} \circ \Phi_{t_2}) = T(\Phi_{t_1+t_2}),$$

and the map $\mathbb{R} \times TM \to TM$, $(t,v) \mapsto \Phi_t(v)$ is smooth (since it is just the restriction of the map $T\Phi : T(\mathbb{R} \times M) \to TM$ to the submanifold $\mathbb{R} \times TM$). Hence, $T\Phi_t$ is a flow on TM, and therefore corresponds to a complete vector field $\hat{X} \in \mathfrak{X}(TM)$. This is called the *tangent lift* of X.

Proposition 5.5. Let $F \in C^{\infty}(M,N)$, and $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ complete vector fields, with flows Φ_t^X , Φ_t^Y .

$$X \sim_F Y \iff F \circ \Phi_t^X = \Phi_t^Y \circ F$$
 for all t .

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In short, vector fields are F-related if and only if their flows are F-related.

Proof. Suppose $F \circ \Phi_t^X = \Phi_t^Y \circ F$ for all *t*. For $g \in C^{\infty}(N)$, and $p \in M$, taking a *t*-derivative of

$$g(F(\boldsymbol{\Phi}_t^X(p))) = g(\boldsymbol{\Phi}_t^Y(F(p)))$$

at t = 0 on both sides, we get

$$(T_p F(X_p))(g) = Y_{F(p)}(g)$$

i.e. $T_pF(X_p) = Y_{F(p)}$. Hence $X \sim_F Y$. Conversely, suppose $X \sim_F Y$. As we had seen, if $\gamma: J \to M$ is a solution curve for X, with initial condition $\gamma(0) = p$ then $F \circ \gamma: J \to M$ is a solution curve for Y, with initial condition F(p). That is, $F(\Phi_t^X(p)) = \Phi_t^Y(F(p))$, or $F \circ \Phi_t^X = \Phi_t^Y \circ F$. \Box

[‡] This generalizes to possibly incomplete vector fields: The vector fields are related if and only if $F \circ \Phi = \Phi \circ (id_{\mathbb{R}} \times F)$. But for simplicity, we only consider the complete case.