1. T^*M has canonical 1-form

$\Theta \in \Omega^1(T^*M)$ defined by $\Theta_{(x,\xi)}(v) = \xi(\pi_* v)$

(here $\xi \in T^*_x M$, $v \in T_{(x,\xi)}(T^*M)$, $\pi : T^*M \to M$.)

- Let $\alpha \in \Omega^1(M)$, which may be viewed as a map $\underline{\alpha} : M \to T^*M$. Prove $\underline{\alpha}^*\Theta = \alpha$.
- Let $(x^1, \dots, x^n)$ be local coords on M, and let $(x^1, \dots, x^n, p_1, \dots, p_n)$ be the induced coordinates on T^*M. Write Θ in these coords.
- prove that $\Omega = d\Theta$ is nondegenerate, hence a symplectic form
  write Ω in local coordinates defined above.
- Let $\mathbb{R}$ act on T^*M via $\rho : \mathbb{R} \times T^*M \longrightarrow T^*M$ $(t, (x, \xi)) \mapsto (x, e^t \xi)$
  compute $E = \rho_*\left(\frac{\partial}{\partial t}\right)$ the Euler vector field.
- Compute $L_E \Omega$. What are possible eigenvalues of L_E ?
- Functions on T^*M : besides basic functions $\pi^* f$, $f \in C^\infty(M, \mathbb{R})$, there are the linear functions : Any vector field $v \in \mathfrak{X}(M)$ defines a function $f_v : (x, \xi) \longmapsto \xi(v(x))$

  If $v = v^i \frac{\partial}{\partial x^i}$, compute f_v in coordinates. What is $L_E f_v$?
- $\Omega^{-1}(df, dg) = \{f, g\}$ defines a Poisson bracket. write $\{f, g\}$ in local coords.

  Compute $\{f_v, \pi^* g\}$ for $v \in \mathfrak{X}(M)$ and $g \in C^\infty(M)$.

  what is $\{f_v, f_w\}$ $v, w \in \mathfrak{X}(M)$?

- Given $H \in C^\infty(T^*M, \mathbb{R})$, the Hamiltonian vector field is
$$X_H = -\Omega^{-1}(dH) \in \mathfrak{X}(T^*M).$$
Prove $[E, X_H] = X_{(L_E H) - H}$.

- Describe the Hamiltonian flow of f_v, $v \in \mathfrak{X}(M)$, in terms of the flow φ_v of v itself.

- Let g be a Riemannian metric, $g^{-1} \in \Gamma(S^2T)$ its inverse, and $f_g : T^*M \longrightarrow \mathbb{R}$ the corresponding quadratic function given by $f_g(x,\xi) = g_x^{-1}(\xi,\xi)$.
Prove X_{f_g} is tangent to the unit sphere bundle $S(T^*M)$
What is $[X_{f_v}, X_{f_g}]$?

- Let $\tilde{\gamma} : \mathbb{R} \to T^*M$ be a trajectory of X_{f_g}, and let $\gamma = \pi_M \circ \tilde{\gamma}$ be the base path. Prove γ is a geodesic.

2. Twisted cotangent bundle. Let $\omega \in \Omega^2(M)$, $d\omega = 0$. The twisted cotangent bundle may be described in two equivalent ways:

A) $T^*_\omega M = T^*M$, but equipped with the symplectic form
$$\Omega + \pi^*\omega$$

B) $T^*_\omega M = \bigsqcup_\alpha T^*U_\alpha \big/ \sim$ for an open cover $\{U_\alpha\}$, where

$$T^*U_{\alpha\beta} \ni (x, \xi) \sim (x, \xi + A_{\alpha\beta}(x)) \in T^*U_{\beta\alpha},$$

where $\omega|_{U_\alpha} = dA_\alpha$ and $A_{\alpha\beta} = A_\alpha - A_\beta \in \Omega^1(U_{\alpha\beta})$.

- Prove $\Omega + \pi^*\omega$ is symplectic.
- Show A) and B) are naturally equivalent.
- If M is prequantized, meaning that there is a $\mathbb{C}$ line bundle $\begin{matrix} U \\ \downarrow \\ M \end{matrix}$ with Hermitian metric h and unitary connection ∇ s.t. $\frac{1}{2\pi i}(F^\nabla) = \omega$, show that $T^*_\omega M$ is also prequantized.
- How is the geodesic flow affected by the addition of ω?
- How is the Poisson bracket affected? What is $[\Omega^{-1}, (\Omega + \pi^*\omega)^{-1}]$?
- An easy way of producing the symplectic S^2 is by symplectic reduction $\mathbb{C}^2 /\!/ S^1$.

  Explain how to construct T^*S^2 and $T^*_\omega S^2$ via symplectic reduction. What if we want to obtain the prequantization via reduction?