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Generalized complex geometry

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Abstract

Generalized complex geometry encompasses complex and symplectic geometry as its extremal special cases. We explore the basic properties of this geometry, including its enhanced symmetry group, elliptic deformation theory, relation to Poisson geometry, and local structure theory. We also define and study generalized complex branes, which interpolate between flat bundles on Lagrangian submanifolds and holomorphic bundles on complex submanifolds.

Contents

Introduction

Generalized complex geometry arose from the work of Hitchin [17] on geometries defined by stable differential forms of mixed degree. Algebraically, it interpolates between a symplectic form ω and a complex structure J by viewing each as a complex (or equivalently, symplectic) structure $\mathcal{J}$ on the direct sum of the tangent and cotangent bundles $T \oplus T^*$, compatible with the natural split-signature metric which exists on this bundle. Remarkably, there is an integrability condition on such generalized complex structures which specializes to the closure of the symplectic form on one hand, and the vanishing of the Nijenhuis tensor of J on the other. This is simply that $\mathcal{J}$ must be integrable with respect to the *Courant bracket*, an extension of the Lie bracket of vector fields to smooth sections of $T \oplus T^*$ which was introduced by Courant and Weinstein [10], [11] in their study of Dirac structures. This bracket may be "twisted" by a closed 3-form H, which may be viewed as the curvature of an S^1 gerbe.

We begin, in Sections 1 and 2, with a review of the natural split-signature orthogonal structure on $T \oplus T^*$ and its associated spin bundle $\wedge^\bullet T^*$, the bundle of differential forms. We then briefly review Dirac geometry and introduce the notion of tensor product of Dirac structures, obtained independently by Alekseev-Bursztyn-Meinrenken in [3].

In Section 3, we treat the basic properties of generalized complex structures. We show that any generalized complex manifold admits almost complex structures and has two natural sets of Chern classes $c_k^\pm$. We show the geometry is determined by a complex pure spinor line subbundle $K \subset \wedge^\bullet T^* \otimes \mathbb{C}$, the *canonical bundle*, which can be seen as the minimal degree component of an induced $\mathbb{Z}$-grading on the H-twisted de Rham complex $(\Omega^\bullet(M), d_H)$. For a complex structure, K is the usual canonical bundle, wheras for a symplectic structure it is generated by the form $e^{i\omega}$. We also describe a real Poisson structure P associated to any generalized complex structure and discuss its modular class. We conclude with an example of a family of generalized complex structures interpolating between a complex and symplectic structure.

In Section 4, we prove a local structure theorem for generalized complex manifolds, analogous to the Darboux theorem in symplectic geometry and the Newlander-Nirenberg theorem in complex geometry. We show that near any regular point for the Poisson structure P, the generalized complex manifold is equivalent, via a generalized symmetry, to a product of a complex space of dimension k (called the *type*) with a symplectic space. Finally, we provide an example whose type is constant outside of a submanifold, along which it jumps to a higher value.

In Section 5, we develop the deformation theory of generalized complex manifolds. It is governed by a differential Gerstenhaber algebra

$$(C^\infty(\wedge^k L^*), d_L, [\cdot,\cdot])$$

constructed from the $+i$-eigenbundle L of $\mathcal{J}$. This differential complex is elliptic, and therefore has finite-dimensional cohomology groups $H^k(M, L)$ over a compact manifold M. Similarly to the case of deformations of complex structure, there is an analytic obstruction map $\Phi : H^2(M, L) \to H^3(M, L)$, and if this vanishes then there is a locally complete family of deformations parametrized by an open set in $H^2(M, L)$. In the case that we are deforming a complex structure, this cohomology group turns out to be

$$H^0(M, \wedge^2\mathcal{T}) \oplus H^1(M, \mathcal{T}) \oplus H^2(M, \mathcal{O}).$$

This is familiar as the "extended deformation space" of Barannikov and Kontsevich [4], for which a geometrical interpretation has been sought for some time.

Finally, in Section 6, we introduce *generalized complex branes*, which are vector bundles supported on certain submanifolds compatible with the generalized complex structure. We show that for a usual symplectic manifold, branes consist not only of flat vector bundles supported on Lagrangians, but also certain bundles over a class of coisotropic submanifolds. These are precisely the co-isotropic A-branes discovered by Kapustin and Orlov [21].

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1. Linear geometry of $V \oplus V^*$: notational conventions

Let V be a real vector space of dimension m and let V^* be its dual space. Then $V \oplus V^*$ is endowed with a natural symmetric bilinear form of signature (m, m), given by

$$\langle X + \xi, Y + \eta\rangle = \tfrac{1}{2}(\xi(Y) + \eta(X)).$$

A general element of its Lie algebra of orthogonal symmetries may be written as a block matrix in the splitting $V \oplus V^*$ via

$$\begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}, \tag{1.1}$$

where $A \in \mathrm{End}(V)$, $B : V \longrightarrow V^*$, $\beta : V^* \longrightarrow V$, and where B and β are skew. Therefore we may view B as a 2-form in $\wedge^2 V^*$ via $B(X) = i_X B$ and similarly we may regard β as an element of $\wedge^2 V$, i.e. a bivector. This corresponds to the observation that $\mathfrak{so}(V \oplus V^*) = \wedge^2(V \oplus V^*) = \mathrm{End}(V) \oplus \wedge^2 V^* \oplus \wedge^2 V$.

By exponentiation, we obtain orthogonal symmetries of $V \oplus V^*$ in the identity component of $\mathrm{SO}(V \oplus V^*)$, as follows:

$$\exp(B) = \begin{pmatrix} 1 & \\ B & 1 \end{pmatrix}, \quad \exp(\beta) = \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}, \quad \exp(A) = \begin{pmatrix} \exp A & \\ & (\exp A^*)^{-1} \end{pmatrix}.$$

The transformations $\exp(B)$ and $\exp(\beta)$ are referred to as B-transforms and β-transforms, respectively, while $\exp(A)$ defines a distinguished embedding of $\mathrm{GL}^+(V)$ into the identity component of the orthogonal group.

1.1. *Maximal isotropics and pure spinors.* We shall be primarily concerned with maximal isotropic subspaces $L \subset V \oplus V^*$, otherwise known as *linear Dirac structures* (see [10]). Examples include the subspaces V, V^*, and $\Delta \oplus \mathrm{Ann}(\Delta)$ for any subspace $\Delta \subset V$. The space of maximal isotropics has two connected components, and elements of these are said to have *even* or *odd* parity, depending on whether they share their connected component with V or not, respectively.

Let $i : \Delta \hookrightarrow V$ be a subspace inclusion and let $\varepsilon \in \wedge^2 \Delta^*$. Then the subspace

$$L(\Delta, \varepsilon) = \{X + \xi \in \Delta \oplus V^* \ : \ i^*\xi = i_X \varepsilon\} \subset V \oplus V^*$$

is an extension of the form

$$0 \longrightarrow \mathrm{Ann}(\Delta) \longrightarrow L(\Delta, \varepsilon) \longrightarrow \Delta \longrightarrow 0$$

and is maximal isotropic; any maximal isotropic may be written in this form. Note that a B-transform preserves projections to V, so it does not affect Δ:

$$\exp(B) \cdot L(\Delta, \varepsilon) = L(\Delta, \varepsilon + i^*B). \tag{1.2}$$

In fact, we see that by choosing B and Δ accordingly, we can obtain any maximal isotropic as a B-transform of $L(\Delta, 0) = \Delta \oplus \mathrm{Ann}(\Delta)$.

Definition 1.1. The *type* of a maximal isotropic $L \subset V \oplus V^*$ is the codimension k of its projection onto V. The parity of L coincides with that of its type.

Maximal isotropic subspaces may alternatively be described by their associated pure spinor lines; we review this relationship in the following. The action of $V \oplus V^*$ on the exterior algebra $\wedge^\bullet V^*$, given by

$$(X + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi, \tag{1.3}$$

extends to a spin representation of the Clifford algebra $\mathrm{CL}(V \oplus V^*)$ associated to the natural inner product $\langle \cdot, \cdot \rangle$. In signature (m, m), the spin representation decomposes according to helicity, and this coincides with the parity decomposition given by

$$\wedge^\bullet V^* = \wedge^{\mathrm{ev}} V^* \oplus \wedge^{\mathrm{od}} V^*.$$

In the spin representation, a general Lie algebra element (1.1) acts in the following way:

$$B \cdot \varphi = -B \wedge \varphi, \qquad \beta \cdot \varphi = i_\beta \varphi, \qquad A \cdot \varphi = -A^* \varphi + \tfrac{1}{2}\mathrm{Tr}(A)\varphi.$$

Exponentiating, we obtain spin group elements corresponding to B-transforms given by

$$\exp(B) \cdot \varphi = e^{-B}\varphi = (1 - B + \tfrac{1}{2} B \wedge B + \cdots) \wedge \varphi \tag{1.4}$$

and similarly for the spinorial action of $\mathrm{GL}^+(V)$, given by

$$\exp(A) \cdot \varphi = g \cdot \varphi = \sqrt{\det g}(g^*)^{-1}\varphi.$$

This indicates that, as a $\mathrm{GL}^+(V)$ representation, the spinors decompose as

$$S = \wedge^\bullet V^* \otimes (\det V)^{1/2}. \tag{1.5}$$

The Spin_0-invariant bilinear form on the above spin representation may be expressed as an invariant bilinear pairing on the forms $\wedge^\bullet V^*$, with values in the determinant line $\det V^*$. This coincides with the Mukai pairing on forms [30] and is given by

$$(s, t) = [s^\top \wedge t]_m, \quad s, t \in \wedge^\bullet V^*,$$

where $s^\top$ denotes the reversal anti-automorphism on forms, and $[\cdot]_m$ is the projection to the component of degree $m = \dim V$. The Mukai pairing is invariant under the identity component of Spin, so that, for example, we have $(\exp B \cdot s, \exp B \cdot t) = (s, t)$, for any $B \in \wedge^2 V^*$.

Definition 1.2. A spinor φ is *pure* when its null space $L_\varphi = \{v \in V \oplus V^* : v \cdot \varphi = 0\}$ is maximal isotropic.

Every maximal isotropic subspace $L \subset V \oplus V^*$ is represented by a unique line $K_L \subset \wedge^\bullet V^*$ of pure spinors, as we now describe. By (1.2), any maximal isotropic $L(\Delta, \epsilon)$ may be expressed as the B-transform of $L(\Delta, 0)$ for B chosen such that $i^*B = -\varepsilon$. The pure spinor line with null space $L(\Delta, 0)$ is precisely $\det(\mathrm{Ann}(\Delta)) \subset \wedge^k V^*$, for k the codimension of $\Delta \subset V$. Hence we recover the following result.

PROPOSITION 1.3 ([9, III.1.9]). *Let $L(\Delta, \varepsilon) \subset V \oplus V^*$ be maximal isotropic, let $(\theta_1, \ldots, \theta_k)$ be a basis for $\mathrm{Ann}(\Delta)$, and let $B \in \wedge^2 V^*$ be a 2-form such that $i^*B = -\varepsilon$, where $i : \Delta \hookrightarrow V$ is the inclusion. Then the pure spinor line K_L representing $L(\Delta, \epsilon)$ is generated by*

$$\varphi = \exp(B)\theta_1 \wedge \cdots \wedge \theta_k. \tag{1.6}$$

Note that φ is of even or odd degree according as L is of even or odd parity.

Exchanging the roles of V and V^*, we see that L may be alternatively described as a β-transform of $\mathrm{Ann}(\Delta') \oplus \Delta'$, for $\Delta' = \pi_{V^*}L$, which has associated pure spinor line $\det(\mathrm{Ann}(L \cap V))$. As a result we obtain the following complement to Proposition 1.3.

Proposition 1.4. *Given a subspace inclusion $i : \Delta \hookrightarrow V$ and a 2-form $B \in \wedge^2 V^*$, there exists a bivector $\beta \in \wedge^2 V$, such that*

$$e^B \det(\mathrm{Ann}(\Delta)) = e^\beta \det(\mathrm{Ann}(L \cap V)),$$

where $L = L(\Delta, -i^*B)$. Note that the image of β in $\wedge^2(V/(L \cap V))$ is unique.

The pure spinor line K_L determined by L forms the beginning of a filtration on spinors

$$K_L = F_0 \subset F_1 \subset \cdots \subset F_m = S, \tag{1.7}$$

where F_k is defined as $\mathrm{CL}^k \cdot K_L$, where CL^k is spanned by products of $\leq k$ generators of the Clifford algebra. Note that $\mathrm{CL}^k \cdot K_L = \mathrm{CL}^m \cdot K_L$ for $k > m$ since L annihilates K_L. This filtration becomes a grading when a maximal isotropic $L' \subset V \oplus V^*$ complementary to L is chosen. Then we obtain a $\mathbb{Z}$-grading on $S = \wedge^\bullet V^*$ of the form (for $\dim V$ even, i.e. $m = 2n$)

$$S = U^{-n} \oplus \cdots \oplus U^n, \tag{1.8}$$

where $U^{-n} = K_L$ and $U^k = (\wedge^{k+n} L') \cdot K_L$, using the inclusion as a subalgebra $\wedge^\bullet L' \subset \mathrm{CL}(V \oplus V^*)$. Furthermore, the Mukai pairing gives a nondegenerate pairing

$$U^{-k} \otimes U^k \longrightarrow \det V^*, \tag{1.9}$$

for all $k = 0, \ldots, n$; hence it respects this alternative grading on forms. Note that $U^n = K_{L'}$, so that the Mukai pairing defines an isomorphism $K_L \otimes K_{L'} \longrightarrow \det V^*$.

2. **The Courant bracket**

Let M be a real m-dimensional smooth manifold. The direct sum of its tangent and cotangent bundles $T \oplus T^*$ is endowed with the same canonical bilinear form we described on $V \oplus V^*$. Therefore, we may view $T \oplus T^*$ as having structure group $\mathrm{SO}(m, m)$. By the action defined in (1.3), the differential forms $\wedge^\bullet T^*$ are a Clifford module for $T \oplus T^*$; the Mukai pairing then becomes a nondegenerate pairing

$$\wedge^\bullet T^* \otimes \wedge^\bullet T^* \longrightarrow \det T^*.$$

We will make frequent use of the correspondence between maximal isotropics in $T \oplus T^*$ and pure spinor lines in $\wedge^\bullet T^*$ to describe structures on $T \oplus T^*$ in terms of differential forms.

The Lie bracket of vector fields may be defined in terms of its interior product with differential forms, via the formula

$$i_{[X,Y]} = [L_X, i_Y] = [[d, i_X], i_Y].$$

Using the spinorial action (1.3) of $T \oplus T^*$ on forms, we may define the *Courant bracket* of sections $e_i \in C^\infty(T \oplus T^*)$ in the same way, following [22]:

$$[e_1, e_2] \cdot \varphi = [[d, e_1 \cdot], e_2 \cdot]\varphi \quad \forall \varphi \in \Omega^\bullet(M). \tag{2.1}$$

The Courant bracket, introduced in [10], [11], is not skew-symmetric, but it follows from the fact that $d^2 = 0$ that the following Jacobi identity holds:

$$[[e_1, e_2], e_3] = [e_1, [e_2, e_3]] - [e_2, [e_1, e_3]]. \tag{2.2}$$

In fact, the Courant bracket defines a Lie algebra up to homotopy [33]. As observed in [35], one may replace the exterior derivative in (2.1) by the twisted operator $d_H\varphi = d\varphi + H \wedge \varphi$ for a real closed 3-form $H \in \Omega^3(M)$. Expanding this expression for $e_1 = X + \xi$ and $e_2 = Y + \eta$, we obtain

$$[X + \xi, Y + \eta]_H = [X, Y] + \mathcal{L}_X\eta - i_Y d\xi + i_X i_Y H. \tag{2.3}$$

The properties of this bracket were used in [27] and [32] to define the concept of *exact Courant algebroid*, which consists of a vector bundle extension

$$0 \longrightarrow T^* \xrightarrow{\pi^*} E \xrightarrow{\pi} T \longrightarrow 0, \tag{2.4}$$

where E is equipped with a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ and bracket $[\cdot, \cdot]$ satisfying the conditions:

C1) $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$,
C2) $\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)]$,
C3) $[e_1, fe_2] = f[e_1, e_2] + (\pi(e_1)f)e_2$, $f \in C^\infty(M)$,
C4) $\pi(e_1)\langle e_2, e_3 \rangle = \langle [e_1, e_2], e_3 \rangle + \langle e_2, [e_1, e_3] \rangle$,
C5) $[e_1, e_1] = \pi^* d\langle e_1, e_1 \rangle$.

Exactness at the middle place forces $\pi^*(T^*)$ to be isotropic, and therefore the inner product on E must be of split signature. It is then always possible to choose an isotropic splitting $s : T \longrightarrow E$ for π, yielding an isomorphism $E \cong T \oplus T^*$ which takes the Courant bracket to that given by (2.3), where H is the curvature of the splitting, i.e.,

$$i_X i_Y H = s^*[s(X), s(Y)], \quad X, Y \in T. \tag{2.5}$$

Isotropic splittings of (2.4) are acted on transitively by the 2-forms $B \in \Omega^2(M)$ via transformations of the form

$$e \mapsto e + \pi^* i_{\pi e} B, \quad e \in E.$$

Such a change of splitting modifies the curvature H by the exact form dB. Hence the cohomology class $[H] \in H^3(M, \mathbb{R})$, called the *Ševera class*, is independent of the splitting and determines the exact Courant algebroid structure on E completely.

2.1. *Symmetries of the Courant bracket.* The Lie bracket of smooth vector fields is invariant under diffeomorphisms; in fact, there are no other symmetries of the tangent bundle preserving the Lie bracket. Since the Courant bracket on $T \oplus T^*$ depends on a 3-form H, it may appear at first glance to have a smaller symmetry group than the Lie bracket. However, as was observed in [35], the spinorial action of 2-forms (1.4) satisfies

$$e^{-B} d_H e^B = d_{H+dB},$$

and therefore we obtain the following action on derived brackets:

$$e^B[e^{-B}\cdot, e^{-B}\cdot]_H = [\cdot,\cdot]_{H+dB}. \tag{2.6}$$

We see from (2.6) that closed 2-forms act as symmetries of any exact Courant algebroid.

Definition 2.1. A B-field transformation is the automorphism of an exact Courant algebroid E defined by a closed 2-form B via

$$e \mapsto e + \pi^* i_{\pi e} B.$$

A diffeomorphism $\varphi : M \longrightarrow M$ lifts to an orthogonal automorphism of $T \oplus T^*$ given by

$$\begin{pmatrix} \varphi_* & 0 \\ 0 & \varphi^{*-1} \end{pmatrix},$$

which we will denote by φ_*. It acts on the Courant bracket via

$$\varphi_*[\varphi_*^{-1}\cdot, \varphi_*^{-1}\cdot]_H = [\cdot,\cdot]_{\varphi^{*-1}H}. \tag{2.7}$$

Combining (2.7) with (2.6), we see that the composition $F = \varphi_* e^B$ is a symmetry of the H-twisted Courant bracket if and only if $\varphi^* H - H = dB$.

Proposition 2.2. *Let F be an orthogonal automorphism of $T \oplus T^*$, covering the diffeomorphism $\varphi : M \longrightarrow M$, and preserving the H-twisted Courant bracket. Then $F = \varphi_* e^B$ for a unique 2-form $B \in \Omega^2(M)$ satisfying $\varphi^* H - H = dB$.*

Proof. Let $G = \varphi_*^{-1} \circ F$, so that it is an automorphism of $T \oplus T^*$ covering the identity satisfying $G[G^{-1}\cdot, G^{-1}\cdot]_H = [\cdot,\cdot]_{\varphi^* H}$. In particular, for any sections $x, y \in C^\infty(T \oplus T^*)$ and $f \in C^\infty(M)$ we have $G[x, fy]_H = [Gx, Gfy]_{\varphi^* H}$, which, using axiom C3), implies

$$\pi(x)(f)Gy = \pi(Gx)(f)Gy.$$

Therefore, $\pi G = \pi$, and so G is an orthogonal map preserving projections to T. This forces it to be of the form $G = e^B$, for B a uniquely determined 2-form. By (2.6), B must satisfy $\varphi^* H - H = dB$. Hence we have $F = \varphi_* e^B$, as required. $\square$

An immediate corollary of this result is that the automorphism group of an exact Courant algebroid E is an extension of the diffeomorphisms preserving the cohomology class $[H]$ by the abelian group of closed 2-forms:

$$0 \longrightarrow \Omega^2_{cl}(M) \longrightarrow \mathrm{Aut}(E) \longrightarrow \mathrm{Diff}_{[H]}(M) \longrightarrow 0.$$

Derivations of a Courant algebroid E are linear first order differential operators D_X on $C^\infty(E)$, covering vector fields X and satisfying

$$X\langle\cdot,\cdot\rangle = \langle D_X\ \cdot,\cdot\rangle + \langle\cdot, D_X\ \cdot\rangle,$$
$$D_X[\cdot,\cdot] = [D_X\ \cdot,\cdot] + [\cdot, D_X\ \cdot].$$

Differentiating a 1-parameter family of automorphisms $F_t = \varphi^t_* e^{B_t}$, $F_0 = \mathrm{Id}$, and using the convention for Lie derivative

$$\mathcal{L}_X = -\tfrac{d}{dt}\Big|_{t=0}\varphi^t_*,$$

we see that the Lie algebra of derivations of the H-twisted Courant bracket consists of pairs $(X,b) \in C^\infty(T) \oplus \Omega^2(M)$ such that $\mathcal{L}_X H = db$. These act via

$$(X,b)\cdot(Y+\eta) = \mathcal{L}_X(Y+\eta) - i_Y b. \tag{2.8}$$

Therefore the algebra of derivations of an exact Courant algebroid algebroid E is an abelian extension of the smooth vector fields by the closed 2-forms:

$$0 \longrightarrow \Omega^2_{cl}(M) \longrightarrow \mathrm{Der}(E) \longrightarrow C^\infty(T) \longrightarrow 0.$$

PROPOSITION 2.3. *Let $D_{X_t} = (X_t, b_t) \in C^\infty(T) \oplus \Omega^2(M)$ be a (possibly time-dependent) derivation of the H-twisted Courant bracket on a compact manifold, so that it satisfies $\mathcal{L}_{X_t} H = db_t$ and acts via* (2.8). *Then it generates a* 1*-parameter subgroup of Courant automorphisms*

$$F^t_{D_X} = \varphi^t_* e^{B_t}, \quad t \in \mathbb{R}, \tag{2.9}$$

where φ^t denotes the flow of the vector field X_t for a time t and

$$B_t = \int_0^t \varphi_s^* b_s \ ds. \tag{2.10}$$

Proof. First we see that $F^t_{D_X}$ is indeed an automorphism, since

$$\begin{aligned} dB_t &= \int_0^t \varphi_s^*(\mathcal{L}_{X_s}H)\ ds = \tfrac{d}{du}\Big|_{u=0}\int_0^t \varphi_s^*\varphi_u^* H\ ds \\ &= \tfrac{d}{du}\Big|_{u=0}\int_u^{t+u} \varphi_{s'}^* H\ ds' \\ &= \varphi_t^* H - H, \end{aligned}$$

which proves the result by Proposition 2.2. To see that it is a 1-parameter subgroup, observe that $e^B\varphi_* = \varphi_* e^{\varphi^* B}$ for any $\varphi \in \mathrm{Diff}(M)$ and $B \in \Omega^2(M)$, so that

$$\varphi_*^{t_1} e^{B_{t_1}} \varphi_*^{t_2} e^{B_{t_2}} = \varphi_*^{t_1+t_2} e^{\varphi_{t_2}^* B_{t_1} + B_{t_2}} = \varphi_*^{t_1+t_2} e^{B_{t_1+t_2}},$$

where we use the expression (2.10) for the final equality. □

It is clear from axioms C1)–C4) that the left adjoint action $\mathrm{ad}_v : w \mapsto [v,w]$ defines a derivation of the Courant algebroid. However, ad is neither surjective nor injective; rather, for E exact, it induces the following exact sequence:

$$0 \longrightarrow \Omega^1_{cl}(M) \xrightarrow{\pi^*} C^\infty(E) \xrightarrow{\mathrm{ad}} \mathrm{Der}(E) \xrightarrow{\chi} H^2(M,\mathbb{R}) \longrightarrow 0, \tag{2.11}$$

where $\Omega^1_{cl}(M)$ denotes the closed 1-forms and we define $\chi(D_X) = [i_X H - b] \in H^2(M,\mathbb{R})$ for $D_X = (X,b)$ as above. Derivations (X,b) in the kernel of χ, namely those for which $b = i_X H + d\xi$ for a 1-form ξ, are called *exact* derivations. A smooth 1-parameter family of automorphisms $F_t = \varphi_*^t e^{B_t}$ from $F_0 = \mathrm{id}$ to $F_1 = F$ is called an *exact* isotopy when it is generated by a smooth time-dependent family of exact derivations.

Definition 2.4. An automorphism $F \in \mathrm{Aut}(E)$ is called *exact* if there is an exact isotopy F_t from $F_0 = \mathrm{id}$ to $F_1 = F$. This defines the subgroup of exact automorphisms of any Courant algebroid:

$$\mathrm{Aut}_{ex}(E) \subset \mathrm{Aut}(E).$$

2.2. *Dirac structures.* The Courant bracket fails to be a Lie bracket due to exact terms involving the inner product $\langle\cdot,\cdot\rangle$. Therefore, upon restriction to a subbundle $L \subset T \oplus T^*$ which is involutive (closed under the Courant bracket) as well as being isotropic, the anomalous terms vanish. Then $(L, [,], \pi)$ defines a Lie algebroid, with associated differential graded algebra $(C^\infty(\wedge^\bullet L^*), d_L)$.

In fact, there is a tight *a priori* constraint on which proper subbundles may be involutive:

Proposition 2.5. *If $L \subset E$ is an involutive subbundle of an exact Courant algebroid, then L must be isotropic, or of the form $\pi^{-1}(\Delta)$, for Δ an integrable distribution in T.*

Proof. Suppose that $L \subset E$ is involutive, but not isotropic; i.e., there exists $v \in C^\infty(L)$ such that $\langle v, v\rangle \neq 0$ at some point $m \in M$. Then for any $f \in C^\infty(M)$,

$$[fv, v] = f[v, v] - (\pi(v)f)v + 2\langle v, v\rangle df,$$

implying that $df|_m \in L|_m$ for all f, i.e. $T^*|_m \subset L|_m$. Since $T^*|_m$ is isotropic, this inclusion must be proper, i.e. $L|_m = \pi^{-1}(\Delta|_m)$, where $\Delta = \pi(L)$ is nontrivial at m. Hence the rank of L must exceed the maximal dimension of an isotropic subbundle. This implies that $T^*|_m \subset L|_m$ at every point m, and hence that Δ is a smooth subbundle of T, which must itself be involutive. Hence $L = \pi^{-1}(\Delta)$, as required. □

Definition 2.6 (Dirac structure). A maximal isotropic subbundle $L \subset E$ of an exact Courant algebroid is called an *almost Dirac* structure. If L is involutive, then the almost Dirac structure is said to be integrable, or simply a *Dirac* structure.

As in Section 1.1, a Dirac structure $L \subset T \oplus T^*$ has a unique description, at a point p, as a generalized graph $L(\Delta, \varepsilon)$, where $\Delta = \pi(L)$ is the projection to T_p and $\varepsilon \in \wedge^2\Delta^*$. Assuming that L is *regular* near p in the sense that Δ has constant rank near p, we have the following description of the integrability condition:

Proposition 2.7. *Let $\Delta \subset T$ be a subbundle and $\varepsilon \in C^\infty(\wedge^2\Delta^*)$. Then the almost Dirac structure $L(\Delta, \varepsilon)$ is integrable for the H-twisted Courant bracket if and only if Δ integrates to a foliation and $d_\Delta\varepsilon = i^*H$, where d_Δ is the leafwise exterior derivative.*

Proof. Let $i : \Delta \hookrightarrow T$ be the inclusion. Then $d_\Delta : C^\infty(\wedge^k\Delta^*) \to C^\infty(\wedge^{k+1}\Delta^*)$ is defined by $i^*\circ d = d_\Delta \circ i^*$. Suppose that $X+\xi, Y+\eta \in C^\infty(L)$, i.e. $i^*\xi = i_X\varepsilon$ and $i^*\eta = i_Y\varepsilon$. Consider the bracket $Z + \zeta = [X + \xi, Y + \eta]$; if L is Courant involutive, then $Z = [X, Y] \in C^\infty(\Delta)$, showing Δ is involutive, and the difference

$$\begin{aligned} i^*\zeta - i_Z\varepsilon &= i^*(\mathcal{L}_X\eta - i_Y d\xi + i_X i_Y H) - i_{[X,Y]}\varepsilon \\ &= d_\Delta i_X i_Y\varepsilon + i_X d_\Delta i_Y\varepsilon - i_Y d_\Delta i_X\varepsilon + i_X i_Y i^*H - [[d_\Delta, i_X], i_Y]\varepsilon \\ &= i_Y i_X(d_\Delta\varepsilon - i^*H) \end{aligned}$$

must vanish for all $X+\xi, Y+\eta \in C^\infty(L)$, showing that $d_\Delta\varepsilon = i^*H$. Reversing the argument we see that the converse holds as well. □

This implies that, in a regular neighbourhood, a d_H-closed generator may always be chosen.

Corollary 2.8. *Let (Δ, ε) be as above and assume $L(\Delta, \varepsilon)$ is integrable; then for $B \in C^\infty(\wedge^2 T^*)$ such that $i^*B = -\varepsilon$, there exists a basis of sections $(\theta_1, \dots, \theta_k)$ for $\mathrm{Ann}(\Delta)$ such that*

$$\varphi = e^B \theta_1 \wedge \cdots \wedge \theta_k$$

is a d_H-closed generator for the pure spinor line K_L.

Proof. Let $\Omega = \theta_1 \wedge \ldots \wedge \theta_k$. By Proposition 2.7, Δ is integrable, so $(\theta_1, \dots, \theta_k)$ can be chosen such that $d\Omega = 0$. Then we have

$$d_H(e^B \Omega) = (dB + H) \wedge e^B \Omega = 0,$$

where the last equality holds since $i^*(dB + H) = -d_\Delta \epsilon + i^*H = 0$. □

In neighbourhoods where Δ is not regular, one may not find d_H-closed generators for K_L; nevertheless, one has the following useful description of the integrability condition.

Theorem 2.9. *The almost Dirac structure $L \subset T \oplus T^*$ is involutive for the H-twisted Courant bracket if and only if*

$$d_H(C^\infty(F_0)) \subset C^\infty(F_1); \tag{2.12}$$

that is, for any local trivialization φ of K_L, there exists a section $X + \xi \in C^\infty(T \oplus T^*)$ such that

$$d_H \varphi = i_X \varphi + \xi \wedge \varphi.$$

Furthermore, Condition (2.12) *implies that*

$$d_H(C^\infty(F_k)) \subset C^\infty(F_{k+1}) \tag{2.13}$$

for all k.

Proof. Let φ be a local generator for $K_L = F_0$. Then for $e_1, e_2 \in C^\infty(L)$, we have

$$\begin{aligned} [e_1, e_2]_H \cdot \varphi &= [[d_H, e_1], e_2]\varphi \\ &= e_1 \cdot e_2 \cdot d_H \varphi, \end{aligned}$$

and therefore L is involutive if and only if $d_H\varphi$ is annihilated by all products $e_1 e_2$, $e_i \in C^\infty(L)$. Since F_k is precisely the subbundle annihilated by products of $k+1$ sections of L, we obtain $d_H \varphi \in F_1$. The subbundle F_1 decomposes in even and odd degree parts as $F_1 = F_0 \oplus (T \oplus T^*) \cdot F_0$, and since d_H is of odd degree, we see that $d_H \varphi \in (T \oplus T^*) \cdot K_L$, as required. To prove (2.13), we proceed by induction on k; let $\psi \in F_k$, then since $[e_1, e_2]_H \cdot \psi = [[d_H, e_1], e_2]\psi$, we have

$$e_1 \cdot e_2 \cdot d_H \psi = d_H(e_1 \cdot e_2 \cdot \psi) + e_1 \cdot d_H(e_2 \cdot \psi) - e_2 \cdot d_H(e_1 \cdot \psi) - [e_1, e_2]_H \cdot \psi.$$

All terms on the right-hand side are in F_{k-1} by induction, implying that $d_H \psi \in F_{k+1}$, as required. □

Since the inner product provides a natural identification $(T \oplus T^*) \cdot K_L = L^* \otimes K_L$, the previous result shows that the pure spinor line generating a Dirac structure is equipped with an operator

$$d_H : C^\infty(K_L) \longrightarrow C^\infty(L^* \otimes K_L), \tag{2.14}$$

which satisfies $d_H^2 = 0$ upon extension to $C^\infty(\wedge^k L^* \otimes K_L)$. This makes K_L a Lie algebroid module for L, i.e. a module over the differential graded Lie algebra $(\wedge^\bullet L^*, d_L)$ associated to the Lie algebroid L (see [14] for a detailed discussion of Lie algebroid modules).

Example 2.10. The cotangent bundle $T^* \subset T \oplus T^*$ is a Dirac structure for any twist $H \in \Omega^3_{cl}(M)$.

Example 2.11. The tangent bundle $T \subset T \oplus T^*$ is a Dirac structure for $H = 0$. Applying any 2-form $B \in \Omega^2(M)$, we see that the graph

$$\Gamma_B = e^B(T) = \{X + i_X B \ : \ X \in T\}$$

is a Dirac structure for $H = dB$.

Example 2.12 (Twisted Poisson geometry). As shown in [35], the graph Γ_β of a bivector field β is a Dirac structure for any H such that $[\beta, \beta] = \wedge^3 \beta^*(H)$. Hence for $H = 0$, β must be Poisson.

Example 2.13 (Foliations). Let $\Delta \subset T$ be a smooth distribution of constant rank. Then $\Delta \oplus \mathrm{Ann}(\Delta) \subset T \oplus T^*$ defines a Dirac structure if and only if Δ is integrable and $H|_\Delta = 0$.

Example 2.14 (Complex geometry). Let $J \in \mathrm{End}(T)$ be an almost complex structure. Then

$$L_J = T_{0,1} \oplus \mathrm{Ann}(T_{0,1}) = T_{0,1} \oplus T^*_{1,0}$$

is integrable if and only if $T_{0,1}$ is involutive and $i^* H = 0$ for the inclusion $i : T^{0,1} \hookrightarrow T \otimes \mathbb{C}$, i.e. H is of type $(1,2) + (2,1)$.

We may apply Theorem 2.9 to give a simple description of the *modular vector field* of a Poisson structure (we follow [13]; for the case of twisted Poisson structures, see [23]). The Dirac structure Γ_β associated to a Poisson structure β has corresponding pure spinor line generated by $\varphi = e^\beta \cdot v$, where $v \in C^\infty(\det T^*)$ is a volume form on the manifold, which we assume to be orientable. By Theorem 2.9, there exists $X + \xi \in C^\infty(T \oplus T^*)$ such that $d\varphi = (X + \xi) \cdot \varphi$. Since L_β annihilates φ, there is a unique $X_v \in C^\infty(T)$, called the *modular vector field* associated to (β, v), such that

$$d\varphi = X_v \cdot \varphi. \tag{2.15}$$

We see from applying d to (2.15) that $\mathcal{L}_{X_v}\varphi = d(X_v \cdot \varphi) + X_v \cdot X_v \cdot \varphi = 0$, implying immediately that X_v is a Poisson vector field (i.e. $[\beta, X_v] = 0$) preserving the volume form v. Of course the modular vector field is not an invariant of the Poisson structure alone; for $f \in C^\infty(M,\mathbb{R})$, one obtains

$$X_{(e^f v)} = X_v + [\beta, f].$$

As a result we see, following Weinstein [37], that X_v defines a class $[X_v]$ in the first Lie algebroid cohomology of Γ_β, called the *modular class* of β:

$$[X_v] \in H^1(M, \Gamma_\beta).$$

2.3. *Tensor product of Dirac structures.* Exact Courant algebroids may be pulled back to submanifolds $\iota : S \hookrightarrow M$. Extending results from [7], we provide a proof in the appendix. It is shown there that if E is an exact Courant algebroid on M, then

$$\iota^* E := K^\perp / K,$$

where $K = \mathrm{Ann}(TS)$ and $K^\perp$ is the orthogonal complement in E, inherits an exact Courant algebroid structure over S, with Ševera class given simply by the pullback along the inclusion. Furthermore, any Dirac structure $L \subset E$ may be pulled back to S via

$$\iota^* L_S := \frac{L \cap K^\perp + K}{K} \subset \iota^* E, \tag{2.16}$$

which is an integrable Dirac structure whenever it is smooth as a subbundle of $\iota^* E$, e.g. if $L \cap K^\perp$ has constant rank on S (see Appendix, Proposition 7.2). We now use this pullback operation to define a Baer sum of Courant algebroids, coinciding with that defined in ([34], [5]).

Definition 2.15. Let E_1, E_2 be exact Courant algebroids over M and let $d : M \longrightarrow M \times M$ be the diagonal embedding. Then we define the *Baer sum* or *tensor product* of E_1 with E_2 to be the exact Courant algebroid (over M)

$$E_1 \boxtimes E_2 = d^*(E_1 \times E_2),$$

which can be written simply as

$$E_1 \boxtimes E_2 = \{(e_1, e_2) \in E_1 \times E_2 \ : \ \pi_1(e_1) = \pi_2(e_2)\} / \{(-\pi_1^* \xi, \pi_2^* \xi) \ : \ \xi \in T^*\},$$

and has Ševera class equal to the sum $[H_1] + [H_2]$.

The standard Courant algebroid $(T \oplus T^*, [\cdot,\cdot]_0)$ acts as an identity element for this operation, and every exact Courant algebroid E has a natural inverse, denoted by $E^\top$, defined as the same Courant algebroid with $\langle \cdot,\cdot \rangle$ replaced with its negative $-\langle \cdot,\cdot \rangle$:

$$E^\top = (E, [\cdot,\cdot], -\langle \cdot,\cdot \rangle, \pi). \tag{2.17}$$

We may now use the Dirac pullback (2.16) to define the tensor product of Dirac structures; an equivalent definition appears in [3].

Definition 2.16. Let $L_1 \subset E_1$, $L_2 \subset E_2$ be Dirac structures and E_1, E_2 as above. We define the *tensor product*

$$L_1 \boxtimes L_2 = d^*(L_1 \times L_2) \subset E_1 \boxtimes E_2,$$

where d^* denotes the Dirac pullback (2.16) by the diagonal embedding. Explicitly, we have

$$L_1 \boxtimes L_2 = (\{(x_1, x_2) \in L_1 \times L_2 \ : \ \pi_1(x_1) = \pi_2(x_2)\} + K)/K, \tag{2.18}$$

where $K = \{(-\pi_1^*\xi, \pi_2^*\xi) \ : \ \xi \in T^*\}$. This is a Dirac structure if it is smooth as a bundle.

Example 2.17. The canonical Dirac structure $T^* \subset E$ acts as a zero element: for any other Dirac structure $L \subset F$, $T^* \boxtimes L = T^* \subset E \boxtimes F$.

Example 2.18. The Dirac structure $\Delta + \mathrm{Ann}(\Delta) \subset T \oplus T^*$ associated to an integrable distribution $\Delta \subset T$ is idempotent:

$$(\Delta + \mathrm{Ann}(\Delta)) \boxtimes (\Delta + \mathrm{Ann}(\Delta)) = \Delta + \mathrm{Ann}(\Delta).$$

Example 2.19. The tensor product of Dirac structures is compatible with B-field transformations:

$$e^{B_1} L_1 \boxtimes e^{B_2} L_2 = e^{B_1 + B_2}(L_1 \boxtimes L_2).$$

Combining this with the previous example, taking $\Delta = T$, we see that Dirac structures transverse to T^* remain so after tensor product. Finally we provide an example where smoothness is not guaranteed.

Example 2.20. Let $L \subset E$ be any Dirac structure, with $L^\top \subset E^\top$ defined by the inclusion $L \subset E$. Then

$$L^\top \boxtimes L = \Delta + \mathrm{Ann}(\Delta) \subset T \oplus T^*,$$

where $\Delta = \pi(L)$. Hence $L^\top \boxtimes L$ is a Dirac structure when $\Delta + \mathrm{Ann}(\Delta)$ is a smooth subbundle, i.e. when Δ has constant rank.

Assuming we choose splittings for E_1, E_2, the tensor product of Dirac structures $L_1 \subset E_1$, $L_2 \subset E_2$ annihilates the *wedge product* $K_1 \wedge K_2$ of the pure spinor lines representing L_1 and L_2. For reasons of skew-symmetry, $K_1 \wedge K_2$ is nonzero only when $L_1 \cap L_2 \cap T^* = \{0\}$. This result also appears in [3]:

PROPOSITION 2.21. *Let L_1, L_2 be Dirac structures in $T \oplus T^*$, and let $\varphi_1 \in K_1$, $\varphi_2 \in K_2$ be (local) generators for their corresponding pure spinor lines in $\wedge^\bullet T^*$. Then*

$$L_1 \boxtimes L_2 \cdot (\varphi_1 \wedge \varphi_2) = 0,$$

and therefore $\varphi_1 \wedge \varphi_2$ is a pure spinor for $L_1 \boxtimes L_2$ as long as $L_1 \cap L_2 \cap T^* = \{0\}$.

Proof. From expression (2.18), we obtain the following simple expression:

$$L_1 \boxtimes L_2 = \{X + \xi + \eta \ : \ X + \xi \in L_1 \text{ and } X + \eta \in L_2\}. \tag{2.19}$$

Then for $X + \xi + \eta \in L_1 \boxtimes L_2$, we have

$$\begin{aligned}&(X + \xi + \eta) \cdot (\varphi_1 \wedge \varphi_2)\\ &\quad = (i_X\varphi_1 + \xi \wedge \varphi_1) \wedge \varphi_2 + (-1)^{|\varphi_1|}\varphi_1 \wedge (i_X\varphi_2 + \eta \wedge \varphi_2) = 0. \qquad \square\end{aligned}$$

The anti-orthogonal map $T \oplus T^* \longrightarrow T \oplus T^*$, given by $X + \xi \mapsto (X+\xi)^\top = X - \xi$, satisfies

$$[(X + \xi)^\top, (Y + \eta)^\top]_H^\top = [X + \xi, Y + \eta]_{-H},$$

so that it takes the Courant algebroid to its inverse (2.17). This operation intertwines with the reversal of forms, in the sense that

$$((X + \xi) \cdot \varphi)^\top = (-1)^{|\varphi|+1}(X + \xi)^\top \cdot \varphi^\top,$$

for any $\varphi \in \wedge^\bullet T^*$, where $|\varphi|$ denotes the degree. As a result, we see that reversal operation on forms corresponds to the reversal $L \mapsto L^\top$ of Dirac structures in $T \oplus T^*$. Since the Mukai pairing of pure spinors φ, ψ, is given by the top degree component of $\varphi^\top \wedge \psi$, we conclude from the perfect pairing (1.9) and Proposition 1.4 that for transverse Dirac structures $L_1, L_2 \subset E$ the tensor product $L_1^\top \boxtimes L_2 \subset T \oplus T^*$ has zero intersection with T and hence is the graph of a Poisson bivector β. This result was first observed in its general form in [3] and is consistent with the appearance of a Poisson structure associated to any Lie bialgebroid in [28].

Proposition 2.22 (Alekseev-Bursztyn-Meinrenken [3]). *Let E be any exact Courant algebroid and $L_1, L_2 \subset E$ be transverse Dirac structures. Then*

$$L_1^\top \boxtimes L_2 = \Gamma_\beta \subset T \oplus T^*,$$

where $\beta \in C^\infty(\wedge^2 T)$ is a Poisson structure.

3. Generalized complex structures

Just as a complex structure may be defined as an endomorphism $J : T \to T$ satisfying $J^2 = -1$ and which is integrable with respect to the Lie bracket, we have the following definition, due to Hitchin [17]:

Definition 3.1. A *generalized complex structure* on an exact Courant algebroid $E \cong T \oplus T^*$ is an endomorphism $\mathcal{J} : E \longrightarrow E$ satisfying $\mathcal{J}^2 = -1$ and which is integrable with respect to the Courant bracket, i.e. its $+i$ eigenbundle $L \subset E \otimes \mathbb{C}$ is involutive.

An immediate consequence of Proposition 2.5 is that the $+i$ eigenbundle of a generalized complex structure must be isotropic, implying that $\mathcal{J}$ must be orthogonal with respect to the natural pairing on E:

PROPOSITION 3.2. *A generalized complex structure $\mathcal{J}$ must be orthogonal and hence defines a symplectic structure $\langle \mathcal{J}\cdot, \cdot\rangle$ on E.*

Proof. Let $x, y \in C^\infty(E)$ and decompose $x = a+\bar{a},\ y = b+\bar{b}$ according to the polarization $E\otimes\mathbb{C} = L\oplus\overline{L}$. Since L must be isotropic by Proposition 2.5,

$$\langle \mathcal{J}x, \mathcal{J}y\rangle = \langle a, \bar{b}\rangle + \langle \bar{a}, b\rangle = \langle x, y\rangle.$$

Hence $\mathcal{J}$ is orthogonal and $\langle \mathcal{J}\cdot, \cdot\rangle$ is symplectic, as required. □

This equivalence between complex and symplectic structures on E compatible with the inner product is illustrated most clearly by examining two extremal cases of generalized complex structures on $T\oplus T^*$. First, consider the endomorphism of $T\oplus T^*$:

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}, \tag{3.1}$$

where J is a usual complex structure on V. Then we see that $\mathcal{J}_J^2 = -1$ and $\mathcal{J}_J^* = -\mathcal{J}_J$. Its $+i$ eigenbundle $L_J = T_{0,1}\oplus T^*_{1,0}$ is, by Example 2.14, integrable if and only if J is integrable and $H^{(3,0)} = 0$.

At the other extreme, consider the endomorphism

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{3.2}$$

where ω is a usual symplectic structure. Again, we observe that $\mathcal{J}_\omega^2 = -1$ and the $+i$ eigenbundle

$$L_\omega = \{X - i\omega(X) \ :\ X \in T\otimes\mathbb{C}\}$$

is integrable, by Example 2.11, if and only if $H = 0$ and $d\omega = 0$.

PROPOSITION 3.3. *Generalized complex manifolds must be even-dimensional.*

Proof. Let $p \in M$ be any point and E_p the fibre of the exact Courant algebroid at p. Let $x \in E_p$ be null, i.e. $\langle x, x\rangle = 0$. Then $\mathcal{J}x$ is also null and is orthogonal to x. Therefore $\{x, \mathcal{J}x\}$ span an isotropic subspace $N \subset E_p$. We may iteratively enlarge the spanning set by adding a pair $\{x', \mathcal{J}x'\}$ for $x' \in N^\perp$, until $N^\perp = N$ and $\dim M = \dim N$ is even. □

At any point $p \in M$, the orthogonal group $O(E_p) \cong O(2n, 2n)$ acts transitively on the space of generalized complex structures at p by conjugation, with stabilizer $U(n,n) = O(2n,2n) \cap \mathrm{GL}(2n,\mathbb{C})$. Therefore the space of generalized complex structures at p is given by the coset space

$$\frac{O(2n,2n)}{U(n,n)}. \tag{3.3}$$

In this sense, a generalized complex structure on an even-dimensional manifold is an integrable reduction of the structure group of E from $O(2n,2n)$ to $U(n,n)$. Since $U(n,n)$ is homotopic to $U(n)\times U(n)$, the $U(n,n)$ structure may be further reduced to $U(n)\times U(n)$, which corresponds geometrically to the choice of a positive definite subbundle $C_+ \subset E$ which is complex with respect to $\mathcal{J}$. The orthogonal complement $C_- = C_+^\perp$ is negative-definite and also complex, and so we obtain the orthogonal decomposition

$$E = C_+ \oplus C_-. \tag{3.4}$$

Note that since $C_\pm$ are definite and $T^* \subset E$ is isotropic, the projection $\pi : C_\pm \to T$ is an isomorphism. Hence we can transport the complex structures on $C_\pm$ to T, obtaining two almost complex structures J_+, J_- on T. Thus we see that a generalized complex manifold must admit an almost complex structure. Furthermore it has two canonically associated sets of Chern classes $c_i^\pm = c_i(T, J_\pm) \in H^{2i}(M,\mathbb{Z})$. Summarizing, and using (3.4), we obtain the following.

Proposition 3.4. *A generalized complex manifold must admit almost complex structures, and has two sets of canonical classes $c_i^\pm \in H^{2i}(M,\mathbb{Z})$ such that the total Chern class*

$$c(E,\mathcal{J}) = c^+ \cup c^-,$$

where $c^\pm = \sum_i c_i^\pm$.

3.1. *Type and the canonical line bundle.* Any exact Courant algebroid has a canonical Dirac structure $T^* \subset E$, and a generalized complex structure $\mathcal{J}$ may be characterized by its action on this Dirac structure, as we now describe.

If $\mathcal{J}T^* = T^*$, then $\mathcal{J}$ determines a usual complex structure on the manifold, and a splitting may be chosen for E so that $\mathcal{J}$ is of the form (3.1). On the other hand, if $\mathcal{J}T^* \cap T^* = \{0\}$, then we have the canonical splitting $E = \mathcal{J}T^* \oplus T^*$, and $\mathcal{J}$ takes the form (3.2), i.e. a symplectic structure.

In general, the subbundle $\mathcal{J}T^* \subset E$ projects to a distribution

$$\Delta = \pi(\mathcal{J}T^*) \subset T \tag{3.5}$$

which may vary in dimension along the manifold. Defining

$$E_\Delta = \frac{T^* + \mathcal{J}T^*}{\mathrm{Ann}(\Delta)},$$

we see that E_Δ is an extension of the form

$$0 \longrightarrow \Delta^* \longrightarrow E_\Delta \xrightarrow{\pi} \Delta \longrightarrow 0,$$

and since $\mathrm{Ann}(\Delta) = T^* \cap \mathcal{J}T^*$ is complex, we see that $\mathcal{J}$ induces a complex structure $\mathcal{J}_\Delta$ on E_Δ such that $\Delta^* \cap \mathcal{J}_\Delta \Delta^* = \{0\}$. Therefore, at each point, Δ inherits a generalized complex structure of symplectic type. Furthermore,

$$\frac{E}{T^* + \mathcal{J}T^*} = T/\Delta,$$

showing that, at each point, T/Δ inherits a complex structure. Ignoring integrability, which we address in the next section, we conclude that a generalized complex manifold carries a canonical symplectic distribution (of variable dimension) with transverse complex structure. At each point, we may choose an isotropic complement $\tilde{\Delta}' \subset E$ to $T^* + \mathcal{J}T^*$, in such a way that $\mathcal{J}\tilde{\Delta}' = \tilde{\Delta}'$. Let $\Delta' = \pi(\tilde{\Delta}')$ be its projection to T, inducing the splittings $T = \Delta \oplus \Delta'$ and

$$E = \tilde{\Delta}' \oplus \mathcal{J}\mathrm{Ann}\Delta' \oplus \mathrm{Ann}\Delta' \oplus \mathrm{Ann}\Delta.$$

This expresses $(E, \mathcal{J})$ as the product of a symplectic structure on $\mathrm{Ann}\Delta' \oplus \mathcal{J}\mathrm{Ann}\Delta'$ and a complex structure on $\tilde{\Delta}' \oplus \mathrm{Ann}\Delta$.

Definition 3.5. The *type* of the generalized complex structure $\mathcal{J}$ is the upper semi-continuous function

$$\mathrm{type}(\mathcal{J}) = \tfrac{1}{2}\dim_{\mathbb{R}} T^* \cap \mathcal{J}T^*,$$

with possible values $\{0, 1, \ldots, n\}$, where $n = \frac{1}{2}\dim_{\mathbb{R}} M$. The type of $\mathcal{J}$ coincides with the type of its $+i$ eigenbundle $L \subset E \otimes \mathbb{C}$, and hence is of fixed parity throughout the manifold.

We have therefore obtained a pointwise local form for a generalized complex structure, depending only on its type.

THEOREM 3.6. *At any point, a generalized complex structure of type k is equivalent, by a choice of isotropic splitting for E, to the direct sum of a complex structure of complex dimension k and a symplectic structure of real dimension $2n - 2k$.*

In real dimension 2, connected generalized complex manifolds must be of constant type 0 or 1, i.e. of symplectic or complex type, whereas in dimension 4, they may be of types 0, 1, or 2, with possible jumping from 0 (symplectic) to 2 (complex) along closed subsets of the manifold; we shall encounter such examples in Sections 4.1 and 5.3.

It also follows from our treatment of Dirac structures that a generalized complex structure is completely characterized by the pure spinor line $K \subset S \otimes \mathbb{C}$ corresponding to the maximal isotropic subbundle L. When a splitting for E

is chosen, we obtain an identification $S = \wedge^\bullet T^* \otimes (\det T)^{1/2}$, and hence K may be viewed as a line subbundle of the complex differential forms. For a symplectic structure, $L_\omega = e^{-i\omega}(T)$, and so

$$K_\omega = e^{-i\omega} \cdot \wedge^0 T^* = \mathbb{C} \cdot e^{i\omega},$$

whereas for a complex structure $L_J = T_{0,1} + T^*_{1,0}$, so that

$$K_J = \wedge^n T^*_{1,0},$$

leading to the following definition.

Definition 3.7. The *canonical line bundle* of a generalized complex structure on $T \oplus T^*$ is the complex pure spinor line subbundle $K \subset \wedge^\bullet T^* \otimes \mathbb{C}$ annihilated by the $+i$ eigenbundle L of $\mathcal{J}$.

Proposition 1.3 states that a generator $\varphi \in K_x$ for the canonical line bundle at the point $x \in M$ must have the form

$$\varphi = e^{B+i\omega}\Omega, \tag{3.6}$$

where $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ for $(\theta_1, \ldots, \theta_k)$ a basis for $L \cap (T^* \otimes \mathbb{C})$, and B, ω are the real and imaginary components of a complex 2-form. As a result we can read off the type of $\mathcal{J}$ at p directly as the least nonzero degree (k) of the differential form φ. The generalized complex structure defines a polarization

$$E \otimes \mathbb{C} = L \oplus \overline{L}, \tag{3.7}$$

and therefore by nondegeneracy of the Mukai pairing from equation (1.9),

$$\langle \varphi, \overline{\varphi} \rangle \neq 0. \tag{3.8}$$

Using (3.6), we obtain

$$\begin{aligned} 0 \neq (e^{B+i\omega}\Omega, e^{B-i\omega}\overline{\Omega}) &= (e^{2i\omega}\Omega, \overline{\Omega}) \\ &= \tfrac{(-1)^{2n-k}(2i)^{n-k}}{(n-k)!}\omega^{n-k} \wedge \Omega \wedge \overline{\Omega}, \end{aligned}$$

which expresses the fact that ω pulls back to the symplectic form on $\Delta = \ker \Omega \wedge \overline{\Omega}$ described earlier, and Ω defines the complex structure transverse to Δ. We also see that $\langle \varphi, \overline{\varphi} \rangle \in \det T^*$ defines an orientation independent of the choice of φ, giving a global orientation on the manifold. This orientation, together with the parity of the type, defines a pair of invariants which distinguish the four connected components of the coset space (3.3).

The canonical line bundle K introduced in this section, along with its complex conjugate $\overline{K}$, are the extremal line bundles of a $\mathbb{Z}$-grading on spinors induced by the generalized complex structure. As described in (1.8), a polarization induces a $\mathbb{Z}$-grading on spinors; therefore a generalized complex structure on $T \oplus T^*$, since it determines a polarization $(T \oplus T^*) \otimes \mathbb{C} = L \oplus \overline{L}$, induces an alternative $\mathbb{Z}$-grading on differential forms

$$\wedge^\bullet T^* \otimes \mathbb{C} = U^{-n} \oplus \cdots \oplus U^n,$$

where $U^n = K$ is the canonical line bundle and $U^{n-k} = \wedge^k \overline{L} \cdot U^n$. Since $\overline{L}$ annihilates U^{-n}, we see that $U^{-n} = \overline{U^n}$ is the canonical line of $-\mathcal{J}$. We therefore have the following convenient description of this $\mathbb{Z}$-grading.

PROPOSITION 3.8. *A generalized complex structure $\mathcal{J}$ on $E = T \oplus T^*$ gives rise to a $\mathbb{Z}$-grading*

$$\wedge^\bullet T^* \otimes \mathbb{C} = U^{-n} \oplus \cdots \oplus U^n,$$

where U^k is the ik-eigenbundle of $\mathcal{J}$ acting in the spin representation, and $U^n = K$.

In the case of a usual complex structure $\mathcal{J}_J$, then the graded components correspond to the well-known (p,q)-decomposition of forms as follows:

$$U^k_J = \bigoplus_{p-q=k} \Omega^{p,q}(M, \mathbb{C}), \tag{3.9}$$

since $\mathcal{J}_J$ acts via the spin representation as J^*, which has eigenvalue $i(p-q)$ on $\Omega^{p,q}$.

The fact that $U^{-n} = \det \overline{L} \cdot U^n$, combined with our previous remark (3.8), implies that

$$U^n \otimes \det L^* \otimes U^n \cong \det T^* \otimes \mathbb{C}.$$

Since the complex bundle L is isomorphic to $(E, \mathcal{J})$, we obtain the following.

COROLLARY 3.9. *The canonical line bundle of a generalized complex manifold has first Chern class satisfying*

$$2c_1(K) = c_1^+ + c_1^-.$$

3.2. *Courant integrability.* The notion of type and the $\mathbb{Z}$-grading on spinors introduced in the last section do not depend on the Courant integrability of the generalized complex structure; they may be associated to any *generalized almost complex* structure:

Definition 3.10. A *generalized almost complex* structure is a complex structure $\mathcal{J}$ on an exact Courant algebroid which is orthogonal with respect to the natural inner product.

Naturally, a generalized almost complex structure $\mathcal{J}$ is said to be integrable when its $+i$ eigenbundle $L \subset E \otimes \mathbb{C}$ is involutive for the Courant bracket, i.e. L is a Dirac structure.

Proposition 3.11. *A generalized complex structure is equivalent to a complex Dirac structure $L \subset E \otimes \mathbb{C}$ such that $L \cap \overline{L} = \{0\}$.*

As a result, $(L, [,], \pi)$, where $\pi : L \longrightarrow T \otimes \mathbb{C}$ is the projection, defines the structure of a Lie algebroid, and therefore we obtain a differential complex

$$C^\infty(\wedge^k L^*) \xrightarrow{d_L} C^\infty(\wedge^{k+1} L^*), \tag{3.10}$$

where d_L is the Lie algebroid de Rham differential, which satisfies $d_L^2 = 0$ due to the Jacobi identity for the Courant bracket restricted to L. The operator d_L has principal symbol $s(d_L) : T^* \otimes \wedge^k L^* \to \wedge^{k+1} L^*$ given by $\pi^* : T^* \to L^*$ composed with wedge product, i.e.

$$s_\xi(d_L) = \pi^*(\xi) \wedge \cdot\ ,$$

where $\xi \in T^*$. We now observe that the complex (3.10) is *elliptic* for a generalized complex structure.

Proposition 3.12. *The Lie algebroid complex of a generalized complex structure is elliptic.*

Proof. Given a real, nonzero covector $\xi \in T^*$, write $\xi = \alpha + \overline{\alpha}$ for $\alpha \in L$. For $v \in L$, we have $\pi^*\xi(v) = \xi(\pi(v)) = \langle \xi, v \rangle = \langle \overline{\alpha}, v \rangle$. Using the inner product to identify $L^* = \overline{L}$, we therefore have $\pi^*\xi = \overline{\alpha}$, which is nonzero if and only if ξ is. As a result, the symbol sequence is exact for any nonzero real covector, as required. $\square$

This provides us with our first invariants associated to a generalized complex structure:

Corollary 3.13. *The cohomology of the complex* (3.10), *called the Lie algebroid cohomology $H^\bullet(M, L)$, is a finite dimensional graded ring associated to any compact generalized complex manifold.*

In the case of a complex structure, $L = T_{0,1} \oplus T^*_{1,0}$, while $d_L = \overline{\partial}$, and so the Lie algebroid complex is a sum of usual Dolbeault complexes, yielding

$$H^k(M, L_J) = \bigoplus_{p+q=k} H^p(M, \wedge^q T_{1,0}).$$

In the case of a symplectic structure, the Lie algebroid L is the graph of $i\omega$, and hence is isomorphic to $T \otimes \mathbb{C}$ as a Lie algebroid. Hence its Lie algebroid cohomology is simply the complex de Rham cohomology.

$$H^k(M, L_\omega) = H^k(M, \mathbb{C}).$$

We now describe a second invariant, obtained from the $\mathbb{Z}$-grading on differential forms induced by $\mathcal{J}$. As we saw in the previous section, a generalized complex structure on $T\oplus T^*$ determines an alternative grading for the differential forms, which may be viewed as the intersection of two complex conjugate filtrations

$$F_i = \oplus_{k=0}^{i} U^{n-k}, \qquad \overline{F_i} = \oplus_{k=0}^{i} U^{-n+k}.$$

More precisely, we have

$$U^k = F_{n-k} \cap \overline{F}_{n+k}. \tag{3.11}$$

By Theorem 2.12, the integrability of $\mathcal{J}$ with respect to $[\cdot,\cdot]_H$ is equivalent to the fact that d_H takes $C^\infty(F_i)$ into $C^\infty(F_{i+1})$. Using (3.11), this happens if and only if d_H takes $C^\infty(U^k)$ into $C^\infty(U^{k-1}\oplus U^k\oplus U^{k+1})$, but since d_H is odd, we see that $\mathcal{J}$ is integrable if and only if $d_H(C^\infty(U^k)) \subset C^\infty(U^{k-1}\oplus U^{k+1})$. Projecting to these two components, we obtain $d_H = \partial + \overline{\partial}$, where

$$C^\infty(U^k) \underset{\overline{\partial}}{\overset{\partial}{\rightleftarrows}} C^\infty(U^{k+1}). \tag{3.12}$$

We state this result in the context of generalized almost complex structures.

Theorem 3.14. *Let $\mathcal{J}$ be a generalized almost complex structure on $T\oplus T^*$, and define*

$$\begin{aligned}\partial &= \pi_{k+1}\circ d_H : C^\infty(U^k) \longrightarrow C^\infty(U^{k+1})\\ \overline{\partial} &= \pi_{k-1}\circ d_H : C^\infty(U^k) \longrightarrow C^\infty(U^{k-1}),\end{aligned}$$

where π_k is the projection onto U^k. Then

$$d_H = \partial + \overline{\partial} + T_L + \overline{T_L}, \tag{3.13}$$

where $T_L \in \wedge^3 L^* = \wedge^3\overline{L}$ is defined by

$$T_L(e_1,e_2,e_3) = \langle [e_1,e_2], e_3\rangle,$$

and acts via the Clifford action in (3.13). *$\mathcal{J}$ is integrable, therefore, if and only if $d_H = \partial+\overline{\partial}$, or equivalently, if and only if*

$$d_H(C^\infty(U^n)) \subset C^\infty(U^{n-1}). \tag{3.14}$$

In the integrable case, since $d_H = \partial + \overline{\partial}$ and $d_H^2 = 0$, we conclude that $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} = -\overline{\partial}\partial$; hence in each direction, (3.12) defines a differential complex.

Remark. Given the above, a generalized complex structure gives rise to a real differential operator $d^{\mathcal{J}} = i(\overline{\partial}-\partial)$, which can also be written $d^{\mathcal{J}} = [d,\mathcal{J}]$, and which satisfies $(d^{\mathcal{J}})^2 = 0$. It is interesting to note that while in the complex case $d^{\mathcal{J}}$ is just the usual d^c-operator $d^c = i(\overline{\partial}-\partial)$, in the symplectic case $d^{\mathcal{J}}$ is equal to the symplectic adjoint of d defined by Koszul [24] and studied by Brylinski [6] in the context of symplectic harmonic forms.

Using the identification $U^{n-k} = \wedge^k L^* \otimes K$ as in (1.8), the operator $\overline{\partial}$ can be viewed as a Lie algebroid connection

$$\overline{\partial} : C^\infty(\wedge^k L^* \otimes K) \to C^\infty(\wedge^{k+1} L^* \otimes K),$$

extended from $d_H : C^\infty(K) \to C^\infty(L^* \otimes K)$ via the rule

$$\overline{\partial}(\mu \otimes s) = d_L \mu \otimes s + (-1)^{|\mu|} \mu \wedge ds, \tag{3.15}$$

for $\mu \in C^\infty(\wedge^k L^*)$ and $s \in C^\infty(K)$, and satisfying $\overline{\partial}^2 = 0$. Therefore K is a module for the Lie algebroid L, and we may call it a *generalized holomorphic bundle*. From the ellipticity of the Lie algebroid complex for L and the fact that K is a module over L, we immediately obtain the following.

PROPOSITION 3.15. *The cohomology of the complex* $(U^\bullet, \overline{\partial})$, *called the* generalized Dolbeault cohomology $H^\bullet_{\overline{\partial}}(M)$, *is a finite dimensional graded module over* $H^\bullet(M, L)$ *associated to any compact generalized complex manifold.*

In the case of a complex structure, equation (3.9) shows that the generalized Dolbeault cohomology coincides with the usual Dolbeault cohomology, with grading

$$H^k_{\overline{\partial}}(M) = \bigoplus_{p-q=k} H^{p,q}_{\overline{\partial}}(M).$$

A special case occurs when the canonical line bundle is *holomorphically trivial*, in the sense that $(K, \overline{\partial})$ is isomorphic to the trivial bundle $M \times \mathbb{C}$ together with the canonical Lie algebroid connection d_L. Then the Lie algebroid complex and the generalized Dolbeault complex $(U^\bullet, \overline{\partial})$ are isomorphic and hence $H^\bullet_{\overline{\partial}}(M) \cong H^\bullet(M, L)$. This holomorphic triviality of K is equivalent to the existence of a nowhere-vanishing section $\rho \in C^\infty(K)$ satisfying $d_H \rho = 0$. In [17], Hitchin calls these generalized Calabi-Yau structures:

Definition 3.16. A *generalized Calabi-Yau structure* is a generalized complex structure with holomorphically trivial canonical bundle, i.e. admitting a nowhere-vanishing d_H-closed section $\rho \in C^\infty(K)$.

An example of a generalized Calabi-Yau structure is of course the complex structure of a Calabi-Yau manifold, which admits a holomorphic volume form Ω trivializing the canonical line bundle. On the other hand, a symplectic structure has canonical line bundle generated by the closed form $e^{i\omega}$, so it too is generalized Calabi-Yau.

Assuming that $c_1(K) = 0$, we may always choose a nonvanishing section $\rho \in C^\infty(K)$; by Theorem 3.14, integrability implies that $d_H \rho = \chi_\rho \cdot \rho$ for a uniquely determined $\chi_\rho \in C^\infty(\overline{L}) = C^\infty(L^*)$. Applying (3.15), we obtain

$$0 = d_H^2 \rho = (d_L \chi_\rho) \cdot \rho - \chi_\rho \cdot (\chi_\rho \cdot \rho),$$

implying that $d_L\chi_\rho = 0$. Just as for the modular class of a Poisson structure (2.15), χ_ρ defines a class in the Lie algebroid cohomology

$$[\chi_\rho] \in H^1(M, L) \tag{3.16}$$

which is the obstruction to the existence of generalized Calabi-Yau structure.

More generally, we may use standard Čech arguments to show that any generalized holomorphic line bundle V is classified up to isomorphism by an element $[V] \in \mathbb{H}^1(\mathcal{L}_{\log})$ in the first hypercohomology of the complex of sheaves $\mathcal{L}_{\log}$, given by

$$C^\infty(\mathbb{C}^*) \xrightarrow{d_L \log} C^\infty(L^*) \xrightarrow{d_L} C^\infty(\wedge^2 L^*) \xrightarrow{d_L} \cdots .$$

Definition 3.17. The Picard group of isomorphism classes of rank 1 generalized holomorphic bundles, i.e. modules over L, is $Pic(\mathcal{J}) = \mathbb{H}^1(\mathcal{L}_{\log})$.

Of course this implies that $\mathcal{J}$ is generalized Calabi-Yau if and only if $[K] = 0$ as a class in $\mathbb{H}^1(\mathcal{L}_{\log})$. The usual exponential map induces a long exact sequence of hypercohomology groups

$$\cdots \longrightarrow H^1(M, L) \longrightarrow \mathbb{H}^1(\mathcal{L}_{\log}) \xrightarrow{c_1} H^2(\mathbb{Z}) \longrightarrow \cdots ,$$

and so we recover the observation (3.16) that when $c_1(K) = 0$ the Calabi-Yau obstruction lies in $H^1(M, L)$.

Example 3.18. Suppose that the complex bundle V is generalized holomorphic for a complex structure $\mathcal{J}_J$. Then the differential $D : C^\infty(V) \longrightarrow C^\infty(L^* \otimes V)$ may be decomposed according to $L = T_{0,1} \oplus T^*_{1,0}$ to yield

$$D = \overline{\partial}_V + \Phi,$$

where $\overline{\partial}_V : C^\infty(V) \longrightarrow C^\infty(T^*_{0,1} \otimes V)$ is a usual partial connection, $\Phi : V \longrightarrow T_{1,0} \otimes V$ is a bundle map, and $D \circ D = 0$ yields the conditions:

- $\overline{\partial}_V^2 = 0$, i.e. V is a usual holomorphic bundle,
- $\overline{\partial}_V(\Phi) = 0$, i.e. Φ is holomorphic,
- $\Phi \wedge \Phi = 0$ in $\wedge^2 T_{1,0} \otimes \mathrm{End}(V)$.

In the rank 1 case, therefore, we obtain the result

$$Pic(\mathcal{J}_J) = H^1(\mathcal{O}^*) \oplus H^0(\mathcal{T}),$$

showing that the generalized Picard group contains the usual Picard group of the complex manifold but also includes its infinitesimal automorphisms.

3.3. *Hamiltonian symmetries.* The Lie algebra $\mathbf{sym}(\omega)$ of infinitesimal symmetries of a symplectic manifold consists of sections $X \in C^\infty(T)$ such that $\mathcal{L}_X\omega = 0$. The *Hamiltonian* vector fields $\mathbf{ham}(\omega)$ are those infinitesimal symmetries generated by smooth functions, in the sense $X = \omega^{-1}(df)$, for $f \in C^\infty(M,\mathbb{R})$. We then have the well-known sequence

$$0 \longrightarrow \mathbf{ham}(\omega) \longrightarrow \mathbf{sym}(\omega) \xrightarrow{\omega^{-1}} H^1(M,\mathbb{R}) \longrightarrow 0.$$

We now give an analogous description of the symmetries of a generalized complex structure and examine the manner in which it specializes in the cases of symplectic and complex geometry.

Definition 3.19. An infinitesimal symmetry $v \in \mathbf{sym}(\mathcal{J})$ of a generalized complex structure $\mathcal{J}$ on the Courant algebroid E is defined to be a section $v \in C^\infty(E)$ which preserves $\mathcal{J}$ under the adjoint action, i.e. $\mathrm{ad}_v \circ \mathcal{J} = \mathcal{J} \circ \mathrm{ad}_v$, or equivalently, $[v, C^\infty(L)] \subset C^\infty(L)$.

In the presence of a generalized complex structure $\mathcal{J}$, a real section $v \in C^\infty(E)$ may be decomposed according to the splitting $E \otimes \mathbb{C} = L \oplus \overline{L}$, yielding $v = v^{1,0} + v^{0,1}$. Clearly $[v^{1,0}, C^\infty(L)] \subset C^\infty(L)$ by the integrability of $\mathcal{J}$. However $[v^{0,1}, C^\infty(L)] \subset C^\infty(L)$ if and only if $d_L v^{0,1} = 0$, where we use the identification $\overline{L} = L^*$. As a result we identify $\mathbf{sym}(\mathcal{J}) = \ker d_L \cap C^\infty(L^*)$, and the differential complex (3.10) provides the following sequence, suggesting the definition of generalized Hamiltonian symmetries:

$$C^\infty(M,\mathbb{C}) \xrightarrow{d_L} \mathbf{sym}(\mathcal{J}) \longrightarrow H^1(M,L) \longrightarrow 0.$$

Definition 3.20. An infinitesimal symmetry $v \in \mathbf{sym}(\mathcal{J})$ is *Hamiltonian*, i.e. $v \in \mathbf{ham}(\mathcal{J})$, when $v = Df$ for $f \in C^\infty(M,\mathbb{C})$, where

$$Df = d_L f + \overline{d_L f} = d(\mathrm{Re} f) - \mathcal{J} d(\mathrm{Im} f).$$

As a result we obtain the following exact sequence of complex vector spaces:

$$0 \longrightarrow \mathbf{ham}(\mathcal{J}) \longrightarrow \mathbf{sym}(\mathcal{J}) \longrightarrow H^1(M,L) \longrightarrow 0.$$

In the case of a symplectic structure, a section $X + \xi \in C^\infty(T \oplus T^*)$ preserves $\mathcal{J}_\omega$ precisely when $\mathcal{L}_X \omega = 0$ and $d\xi = 0$. On the other hand, computing Df, we obtain

$$Df = d(\mathrm{Re} f) + \omega^{-1} d(\mathrm{Im} f),$$

showing that $X + \xi$ is Hamiltonian precisely when X is Hamiltonian and ξ is exact.

In the complex case, $X + \xi$ preserves $\mathcal{J}_J$ exactly when $\overline{\partial}(X^{1,0} + \xi^{0,1}) = 0$, i.e. when X is a holomorphic vector field and $\overline{\partial}\xi^{0,1} = 0$. We also have

$$Df = \overline{\partial} f + \partial \bar{f},$$

showing that $X + \xi$ is Hamiltonian exactly when $X = 0$ and $\xi = \overline{\partial} f + \partial \bar{f}$ for $f \in C^\infty(M,\mathbb{C})$.

Even for a usual complex manifold, therefore, there are nontrivial Hamiltonian symmetries $\xi = \overline{\partial} f + \partial \overline{f}$, which integrate to B-field transformations e^{tB}, for $B = \partial\overline{\partial}(f - \overline{f})$.

3.4. *The Poisson structure and its modular class.* The symplectic distribution $\Delta = \pi(\mathcal{J}T^*)$ associated to a generalized complex structure is the image of a natural bivector field defined by $\mathcal{J}$:

$$P = \pi \circ \mathcal{J}|_{T^*} : \xi \mapsto \pi(\mathcal{J}\xi). \tag{3.17}$$

It is natural to ask, therefore, whether P is Poisson. For a direct investigation of the properties of P, see [26], [12], and [1]. We present a Dirac-geometric proof that P defines a Poisson structure.

PROPOSITION 3.21. *Let $\mathcal{J}$ be a generalized complex structure with $+i$ eigenbundle $L \subset E \otimes \mathbb{C}$. Then*

$$L^\top \boxtimes \overline{L} = \Gamma_{iP/2}, \tag{3.18}$$

i.e. the tensor product of $L^\top$ with $\overline{L}$ is the graph of the bivector field $iP/2$ in the Courant algebroid $(T \oplus T^*) \otimes \mathbb{C}$, and therefore P must be Poisson.

Proof. Choose a splitting for the Courant algebroid, and express $\mathcal{J}$ as the block matrix

$$\mathcal{J} = \begin{pmatrix} A & P \\ \sigma & -A^* \end{pmatrix}, \tag{3.19}$$

so that the Poisson tensor P is apparent. Let $\zeta \in T^* \otimes \mathbb{C}$, so that $\zeta - i\mathcal{J}\zeta \in L$, or, using (3.19), we have $\zeta - iP\zeta + iA^*\zeta \in L$. Therefore $(\zeta - iP\zeta + iA^*\zeta)^\top = (-\zeta - iP\zeta - iA^*\zeta) \in L^\top$ and $\zeta + iP\zeta - iA^*\zeta \in \overline{L}$. Combining these using (2.19), we see that

$$iP\zeta + 2\zeta \in L^\top \boxtimes \overline{L}$$

and hence $\Gamma_{iP/2} \subset L^\top \boxtimes \overline{L}$. Since both sides are maximal isotropic subbundles, we must have equality, as required. ☐

COROLLARY 3.22. *The distribution $\Delta = \pi(\mathcal{J}T^*) = \mathrm{Im}P$ integrates to a generalized foliation by smooth symplectic leaves with codimension $2k$, where $k = \mathrm{type}(\mathcal{J})$.*

We now observe that this implies a relation between the Calabi-Yau obstruction class and the modular class.

PROPOSITION 3.23. *Let $\mathcal{J}$ be a generalized complex structure such that $c_1(K) = 0$, and let $\rho \in C^\infty(K)$ be a nonvanishing section with*

$$d_H\rho = v \cdot \rho, \quad v \in C^\infty(E).$$

Then $-2\pi(\mathcal{J}v) = X$ is the modular vector field associated to the Poisson structure P and volume form $(\rho, \overline{\rho})$.

Proof. Let $d_H\rho = v^{0,1} \cdot \rho$ for uniquely defined $v^{0,1} \in C^\infty(\overline{L})$, so that $v = v^{1,0} + v^{0,1}$ for $v^{1,0} = \overline{v^{0,1}}$. By equation (3.18) and Proposition 2.21, we have that

$$\rho^\top \wedge \overline{\rho} = e^{\frac{iP}{2}}(\rho, \overline{\rho}) = \varphi.$$

Taking the exterior derivative, and using the definition (2.15) of the modular vector field, we have

$$\begin{aligned} d\varphi = \widetilde{X} \cdot \varphi &= (-1)^{|\rho|}((d_H\rho)^\top \wedge \overline{\rho} + \rho^\top \wedge (d_H\overline{\rho})) \\ &= (-1)^{|\rho|}((v^{0,1} \cdot \rho)^\top \wedge \overline{\rho} + \rho^\top \wedge (v^{1,0} \cdot \overline{\rho})) \\ &= -\pi(v^{0,1} - v^{1,0}) \cdot (\rho^\top \wedge \overline{\rho}) \\ &= -i\pi(\mathcal{J}v) \cdot \varphi, \end{aligned}$$

showing that $\widetilde{X} = -i\pi(\mathcal{J}v)$ is the modular vector field for $iP/2$. Rescaling the Poisson structure, we obtain the result. $\square$

Corollary 3.24. *The Poisson structure P associated to a generalized Calabi-Yau manifold is* unimodular *in the sense of Weinstein* [37], *i.e. it has vanishing modular class.*

The map $H^1(M, L) \longrightarrow H^1(M, \Gamma_P)$ of Lie algebroid cohomology groups implicit in the above result may be understood from the fact that the projection map $T^* \otimes \mathbb{C} \longrightarrow L$ obtained from the splitting $E \otimes \mathbb{C} = L \oplus \overline{L}$, is actually a Lie algebroid morphism, when $T^* \otimes \mathbb{C}$ is endowed with the Poisson Lie algebroid structure, as we now explain.

Proposition 3.25. *Let L, P be the $+i$-eigenbundle and Poisson structure associated to a generalized complex structure. The bundle map $a : \Gamma_P \otimes \mathbb{C} \longrightarrow L$ given, for any $\xi \in T^* \otimes \mathbb{C}$, by*

$$a : \xi + P\xi \mapsto i\xi + \mathcal{J}\xi, \tag{3.20}$$

is a Lie algebroid homomorphism.

Proof. The map a commutes with the projections to the tangent bundle, since $P = \pi \circ \mathcal{J}|_{T^*}$. Given 1-forms ξ, η, we have

$$\begin{aligned} [a(\xi + P\xi), a(\eta + P\eta)] &= i([\xi, \mathcal{J}\eta] + [\mathcal{J}\xi, \eta]) + [\mathcal{J}\xi, \mathcal{J}\eta] \\ &= i([\xi, P\eta] + [P\xi, \eta]) + \mathcal{J}([\xi, P\eta] + [P\xi, \eta]) \\ &= a([\xi + P\xi, \eta + P\eta]), \end{aligned}$$

as required. $\square$

As a final example of the relationship between a generalized complex structure and its associated Poisson structure, we use the above Lie algebroid homomorphism to relate the infinitesimal symmetries of each structure.

PROPOSITION 3.26. *If $\mathcal{J}$ is a generalized complex structure and P its associated Poisson structure, then the maps $E \longrightarrow T$ defined by $v \mapsto \pi(v)$ and $v \mapsto \pi(\mathcal{J}v)$ both induce homomorphisms*

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{ham}(\mathcal{J}) \arrow[r] \arrow[d] & \mathbf{sym}(\mathcal{J}) \arrow[r] \arrow[d] & H^1(M, L) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbf{ham}(P) \arrow[r] & \mathbf{sym}(P) \arrow[r] & H^1(M, \Gamma_P) \arrow[r] & 0 \end{tikzcd}$$

from the infinitesimal symmetries of $\mathcal{J}$ to the infinitesimal symmetries of P.

Proof. Identifying $\mathbf{sym}(\mathcal{J}) = \ker d_L \cap C^\infty(L^*)$, we see from Proposition 3.25 that $a^* : L^* \longrightarrow \Gamma_P^* \otimes \mathbb{C}$ is a morphism of differential complexes. Identifying $\Gamma_P^* \cong T$, and taking real and imaginary parts, we obtain morphisms $v \mapsto \pi(v)$, $v \mapsto \pi(\mathcal{J}v)$ as required. □

3.5. *Interpolation.* Let M be a real manifold of dimension $4k$ with complex structure I and holomorphic symplectic structure $\sigma = \omega_J + i\omega_K$, so that σ is a nondegenerate closed $(2,0)$-form. Since ω_J is of type $(2,0)+(0,2)$, we have $\omega_J I = I^*\omega_J$, and hence

$$\begin{pmatrix} & -\omega_J^{-1} \\ \omega_J & \end{pmatrix}\begin{pmatrix} -I & \\ & I^* \end{pmatrix} = -\begin{pmatrix} -I & \\ & I^* \end{pmatrix}\begin{pmatrix} & -\omega_J^{-1} \\ \omega_J & \end{pmatrix};$$

that is, the generalized complex structures $\mathcal{J}_{\omega_J}$ and $\mathcal{J}_I$ *anticommute*. Hence we may form the one-parameter family of generalized almost complex structures

$$\mathcal{J}_t = (\sin t)\mathcal{J}_I + (\cos t)\mathcal{J}_{\omega_J}, \quad t \in [0, \tfrac{\pi}{2}].$$

Clearly $\mathcal{J}_t$ is a generalized almost complex structure; we now check that it is integrable[1].

PROPOSITION 3.27. *Let M be a holomorphic symplectic manifold as above. Then the generalized almost complex structure $\mathcal{J}_t = (\sin t)\mathcal{J}_I + (\cos t)\mathcal{J}_{\omega_J}$ is integrable for all $t \in [0, \frac{\pi}{2}]$. Therefore it is a family of generalized complex structures interpolating between a symplectic structure and a complex structure.*

[1]An equivalent example may be found in [17].

Proof. Let $B = (\tan t)\omega_K$, a closed 2-form which is well defined for all $t \in [0, \frac{\pi}{2})$. Noting that $\omega_K I = I^*\omega_K = \omega_J$, we obtain the following expression:

$$e^B \mathcal{J}_t e^{-B} = \begin{pmatrix} 0 & -((\sec t)\omega_J)^{-1} \\ (\sec t)\omega_J & 0 \end{pmatrix}.$$

We conclude from this that for all $t \in [0, \frac{\pi}{2})$, $\mathcal{J}_t$ is a B-field transform of the symplectic structure determined by $(\sec t)\omega_J$, and is therefore integrable as a generalized complex structure; at $t = \frac{\pi}{2}$, $\mathcal{J}_t$ is purely complex, and is integrable by assumption, completing the proof. □

4. **Local structure: the generalized Darboux theorem**

The Newlander-Nirenberg theorem informs us that an integrable complex structure on a $2n$-manifold is locally equivalent, via a diffeomorphism, to $\mathbb{C}^n$. Similarly, the Darboux theorem states that a symplectic structure on a $2n$-manifold is locally equivalent, via a diffeomorphism, to the standard symplectic structure $(\mathbb{R}^{2n}, \omega_0)$, where in coordinates $(x_1, \ldots, x_n, p_1, \ldots, p_n)$,

$$\omega_0 = dx_1 \wedge dp_1 + \cdots + dx_n \wedge dp_n.$$

In this section we prove an analogous theorem for generalized complex manifolds, describing a local normal form for a *regular neighbourhood* of a generalized complex manifold.

Definition 4.1. A point $p \in M$ in a generalized complex manifold is called *regular* when the Poisson structure P is regular at p, i.e. $\mathrm{type}(\mathcal{J})$ is locally constant at p.

By Corollary 3.22, a generalized complex structure defines, in a regular neighbourhood U, a foliation $\mathcal{F}$ by symplectic leaves of codimension $2k = 2\,\mathrm{type}(\mathcal{J})$, integrating the distribution $\Delta = \pi(\mathcal{J}T^*)$. The complex structure transverse to Δ described in Section 3.1 defines an integrable complex structure on the leaf space $U/\mathcal{F}$ as we now describe.

Proposition 4.2. *The leaf space $U/\mathcal{F}$ of a regular neighbourhood of a generalized complex manifold inherits a canonical complex structure.*

Proof. Let $L \subset E$ be the $+i$-eigenbundle of $\mathcal{J}$ and let $D = \pi_{T\otimes\mathbb{C}}(L)$ be its projection to the complex tangent bundle, which is smooth in a regular neighbourhood since $\mathrm{type}(\mathcal{J}) = \dim L \cap (T^* \otimes \mathbb{C})$. Then $E \otimes \mathbb{C} = L \oplus \overline{L}$ implies that $T \otimes \mathbb{C} = D + \overline{D}$, while $D \cap \overline{D} = \Delta \otimes \mathbb{C}$. Since the projection π is bracket-preserving, we see that D is an integrable distribution, hence $[\Delta, D] \subset D$. This implies that D descends to an integrable subbundle $D' \subset T(U/\mathcal{F}) \otimes \mathbb{C}$ satisfying $D' \cap \overline{D}' = \{0\}$, hence defining an integrable complex structure on

$U/\mathcal{F}$, as required. This coincides with the complex structure induced by $\mathcal{J}$ on $E/(T^* + \mathcal{J}T^*) = T/\Delta$. $\square$

THEOREM 4.3 (Generalized Darboux theorem). *A regular point of type k in a generalized complex manifold has a neighbourhood which is equivalent to the product of an open set in $\mathbb{C}^k$ with an open set in the standard symplectic space $(\mathbb{R}^{2n-2k}, \omega_0)$.*

Proof. Let U be a regular neighbourhood. Proposition 4.2 guarantees the existence of holomorphic coordinates $(z_1, \ldots, z_k)$ transverse to the symplectic foliation in U. By Weinstein's normal form for regular Poisson structures [36], we can find a leaf-preserving local diffeomorphism $\varphi : \mathbb{R}^{2n-2k} \times \mathbb{C}^k \longrightarrow U$ such that the leafwise symplectic structure pulls back to $\omega_0 = dx_1 \wedge dp_1 + \cdots + dx_{n-k} \wedge dp_{n-k}$, for x_i, p_i standard coordinates in $\mathbb{R}^{2n-2k}$. We now choose a splitting of the Courant algebroid E over U such that the generalized complex structure $\mathcal{J}$ is a simple product. Let $\Delta \subset TU$ denote the symplectic distribution, while $\Delta' \subset TU$ is the complement induced by φ. Choose an isotropic subbundle $\widetilde{\Delta}' \subset E$, projecting isomorphically to Δ', such that $\mathcal{J}(\widetilde{\Delta}') = \tilde{\Delta}'$ (this choice is locally unobstructed). This defines an isotropic splitting of E, and an isomorphism

$$
\begin{aligned}
E &= \mathcal{J}(\mathrm{Ann}\Delta') \oplus \tilde{\Delta}' \oplus \mathrm{Ann}\Delta' \oplus \mathrm{Ann}\Delta \\
&\cong \Delta \oplus \Delta' \oplus \mathrm{Ann}\Delta' \oplus \mathrm{Ann}\Delta,
\end{aligned} \tag{4.1}
$$

identifying $\mathcal{J}$ with the product of the given complex structure on $\Delta' \oplus \mathrm{Ann}\Delta$ and the given symplectic structure ω_0 on $\Delta \oplus \mathrm{Ann}\Delta'$. Indeed, the $+i$-eigenbundle of $\mathcal{J}$ may be written as

$$L = e^{i\omega_0}(\Delta) \oplus \Delta'_{0,1} \oplus (\Delta')^*_{1,0}. \tag{4.2}$$

While we have expressed $\mathcal{J}$ as a product, the splitting of E given by (4.1) is not necessarily involutive; let $H \in \Omega^3(U, \mathbb{R})$ be the associated 3-form, as defined in (2.5). What remains to show is that the splitting may be modified to an involutive one, while preserving the form of $\mathcal{J}$. For this, we use the tri-grading of forms induced by the product structure $U = \mathbb{R}^{2n-2k} \times \mathbb{C}^k$: differential forms now have degree (p, q, r) for components in $\Omega^p(\mathbb{R}^{2n-2k}) \otimes \Omega^{q,r}(\mathbb{C}^k)$, and the de Rham operator decomposes into a sum of three operators

$$d = d_\Delta + \partial + \overline{\partial}.$$

The involutivity of L, together with (4.2), imply that H has degree $(1,1,1)$ + $(0,2,1) + (0,1,2)$, with both H^{111} and $H^{021+012}$ being real. Since $dH = 0$, it follows that $d_\Delta H^{111} = 0$, so by the Poincaré lemma, there exists a real 2-form $B_1 \in \Omega^{011}(U)$ such that $d_\Delta B_1 = H^{111}$. Hence $H' = H - dB_1$ is of type $(0,2,1) + (0,1,2)$. By the Dolbeault lemma, there exists another real 2-form B_2^{011} with $H' - dB_2 = 0$. Hence, changing the splitting of E by the real

2-form $B_1 + B_2$, we obtain an involutive splitting. However, note that since $B_1 + B_2 \in \Omega^{011}(U)$, this change has no effect on the expression for $\mathcal{J}$, as can be seen from (4.2).

Therefore, we obtain a local isomorphism between E and the standard Courant bracket, taking $\mathcal{J}$ to a product of a complex structure of dimension k with a symplectic structure, as required. $\square$

4.1. *Type jumping.* While Theorem 4.3 fully characterizes generalized complex structures in regular neighbourhoods, it remains an essential feature of the geometry that the *type* of the structure may vary throughout the manifold. The most generic type is zero, when there are only symplectic directions and the Poisson structure P has maximal rank. The type may jump up along closed subsets, has maximal value $n = \frac{1}{2}\dim_{\mathbb{R}} M$, and has fixed parity throughout the manifold. We now present a simple example of a generalized complex structure on $\mathbb{R}^4$ which is of symplectic type ($k = 0$) outside a codimension 2 surface and jumps up to complex type ($k = 2$) along the surface.

Consider the differential form

$$\rho = z_1 + dz_1 \wedge dz_2, \tag{4.3}$$

where z_1, z_2 are the standard coordinates on $\mathbb{C}^2 \cong \mathbb{R}^4$. Along $z_1 = 0$, we have $\rho = dz_1 \wedge dz_2$ and so it generates the pure spinor line corresponding to the standard complex structure. Whenever $z_1 \neq 0$, ρ may be rewritten as follows:

$$\rho = z_1 e^{\frac{dz_1 \wedge dz_2}{z_1}}.$$

Therefore away from $z_1 = 0$, ρ generates the canonical line bundle of the B-field transform of the symplectic form ω, where

$$B + i\omega = z_1^{-1} dz_1 \wedge dz_2.$$

Hence, algebraically the form ρ defines a generalized almost complex structure which is generically of type 0 but jumps to type 2 along $z_1 = 0$.

To verify the integrability of this structure, we differentiate, obtaining $d\rho = (-\partial_{z_2}) \cdot \rho$, showing that ρ indeed satisfies the integrability condition of Theorem 3.14, and defines a generalized complex structure on all of $\mathbb{R}^4$. In this case it is easy to see that although the canonical line bundle is topologically trivial, it does not admit a closed, nowhere-vanishing section. Hence the generalized complex structure is not generalized Calabi-Yau.

In the next chapter we will produce more general examples of the jumping phenomenon. However we indicate here that the simple example above was used in [8] to produce, via a surgery on a symplectic 4-manifold, an example of a compact, simply-connected generalized complex 4-manifold which admits neither complex nor symplectic structures.

5. **Deformation theory**

In the deformation theory of complex manifolds developed by Kodaira, Spencer, and Kuranishi, one begins with a compact complex manifold (M, J) with holomorphic tangent bundle $\mathcal{T}$, and constructs an analytic subvariety $\mathcal{Z} \subset H^1(M, \mathcal{T})$ (containing 0) which is the base space of a family of deformations $\mathcal{M} = \{\epsilon(z) \ : \ z \in \mathcal{Z}, \ \epsilon(0) = 0\}$ of the original complex structure J. This family is *locally complete* (also called *miniversal*), in the sense that any family of deformations of J can be obtained, up to equivalence, by pulling $\mathcal{M}$ back by a map f to $\mathcal{Z}$, as long as the family is restricted to a sufficiently small open set in its base.

The subvariety $\mathcal{Z} \subset H^1(M, \mathcal{T})$ is defined as the zero set of a holomorphic map $\Phi : H^1(M, \mathcal{T}) \to H^2(M, \mathcal{T})$, and so the base of the miniversal family is certainly smooth when this obstruction map vanishes.

In this section we extend these results to the generalized complex setting, following the method of Kuranishi [25]. In particular, we construct, for any generalized complex manifold, a locally complete family of deformations. We then proceed to produce new examples of generalized complex structures by deforming known ones.

5.1. *Lie bialgebroids and the deformation complex.* The generalized complex structure $\mathcal{J}$ on the exact Courant algebroid E is determined by its $+i$-eigenbundle $L \subset E \otimes \mathbb{C}$ which is isotropic, satisfies $L \cap \overline{L} = \{0\}$, and is Courant involutive. Recall that since $E \otimes \mathbb{C} = L \oplus \overline{L}$, we use the natural metric $\langle , \rangle$ to identify $\overline{L}$ with L^*.

To deform $\mathcal{J}$ we will vary L in the Grassmannian of maximal isotropics. Any maximal isotropic having zero intersection with $\overline{L}$ (this is an open set containing L) can be uniquely described as the graph of a homomorphism $\epsilon : L \longrightarrow \overline{L}$ satisfying $\langle \epsilon X, Y \rangle + \langle X, \epsilon Y \rangle = 0$ for all $X, Y \in C^\infty(L)$, or equivalently $\epsilon \in C^\infty(\wedge^2 L^*)$. Therefore the new isotropic is given by

$$L_\epsilon = (1 + \epsilon)L = \{u + i_u \epsilon \ : \ u \in L\}.$$

As the deformed $\mathcal{J}$ is to remain real, we must have $\overline{L}_\epsilon = (1 + \overline{\epsilon})\overline{L}$. Now we observe that L_ϵ has zero intersection with its conjugate if and only if the endomorphism we have described on $L \oplus L^*$, namely

$$A_\epsilon = \begin{pmatrix} 1 & \overline{\epsilon} \\ \epsilon & 1 \end{pmatrix}, \tag{5.1}$$

is invertible; this is the case for ϵ in an open set around zero.

So, providing ϵ is small enough, $\mathcal{J}_\epsilon = A_\epsilon \mathcal{J} A_\epsilon^{-1}$ is a new generalized almost complex structure, and all nearby almost structures are obtained in this way. Note that while A_ϵ itself is not an orthogonal transformation, of course $\mathcal{J}_\epsilon$ is.

To describe the condition on $\epsilon \in C^\infty(\wedge^2 L^*)$ which guarantees that $\mathcal{J}_\epsilon$ is integrable, we observe the following. Since $L^* = \overline{L}$, we have not only an elliptic differential complex (by Proposition 3.12)

$$(C^\infty(\wedge^\bullet L^*), d_L)\,,$$

but also a Lie algebroid structure on L^* coming from the Courant bracket on $\overline{L}$. In fact, by a theorem of Liu-Weinstein-Xu [27], the differential is a derivation of the bracket and we obtain the structure of a Lie bialgebroid in the sense of Mackenzie-Xu [28], also known as a differential Gerstenhaber algebra.

Theorem 5.1 ([27, Th. 2.6]). *Let E be an exact Courant algebroid and $E = L \oplus L'$ for Dirac structures L, L'. Then $L' = L^*$ using the inner product, and the dual pair of Lie algebroids (L, L^*) defines a Lie bialgebroid, i.e.*

$$d_L[a, b] = [d_L a, b] + [a, d_L b],$$

for $a, b \in C^\infty(L^*)$, where $[\cdot,\cdot]$ is extended in the Schouten sense to $C^\infty(\wedge^\bullet L^*)$. Therefore the data

$$(C^\infty(\wedge^\bullet L^*), d_L, [\cdot,\cdot])$$

define a differential Gerstenhaber algebra.

Interpolating between Examples 2.11 and 2.12, Liu-Weinstein-Xu [27] also prove that, under the assumptions of the previous theorem, the graph L_ϵ of a section $\epsilon \in C^\infty(\wedge^2 L^*)$ defines an integrable Dirac structure if and only if it satisfies the Maurer-Cartan equation.

Theorem 5.2 ([27, Th. 6.1]). *The almost Dirac structure L_ϵ, for $\epsilon \in C^\infty(\wedge^2 L^*)$, is integrable if and only if ϵ satisfies the Maurer-Cartan equation*

$$d_L\epsilon + \tfrac{1}{2}[\epsilon, \epsilon] = 0. \tag{5.2}$$

Here $d_L : C^\infty(\wedge^k L^*) \to C^\infty(\wedge^{k+1} L^*)$ and $[\cdot,\cdot]$ is the Lie algebroid bracket on L^*.

Therefore we conclude that the deformed generalized almost complex structure $\mathcal{J}_\epsilon$ is integrable if and only if ϵ satisfies the Maurer-Cartan equation (5.2). We may finally define a smooth family of deformations of the generalized complex structure $\mathcal{J}$. We are only interested in "small" deformations.

Definition 5.3. Let U be an open ball containing the origin of a finite-dimensional vector space. A *smooth family of deformations* of $\mathcal{J}$ over U is a family of sections $\epsilon(u) \in C^\infty(\wedge^2 L^*)$, smoothly varying in $u \in U$, with $\epsilon(0) = 0$, such that (5.1) is invertible and satisfying the Maurer-Cartan equation (5.2) for each $u \in U$. Two such families $\epsilon_1(u), \epsilon_2(u)$ are *equivalent* if $F_u(L_{\epsilon_1}(u)) = L_{\epsilon_2}(u)$ for all $u \in U$, where F_u is a smooth family of Courant automorphisms with $F_0 = \mathrm{id}$.

The space of solutions to (5.2) is infinite-dimensional, however due to the action of the group of Courant automorphisms we are able, as in the case of complex manifolds, to take a suitable quotient, forming a finite-dimensional locally complete family. To obtain this finite-dimensional moduli space of deformations, it will suffice to consider equivalences F_u which are families of exact Courant automorphisms in the sense of Definition 2.4, generated by *time-independent* derivations $\mathrm{ad}(v(u))$ given by a smooth family of sections $v(u) \in C^\infty(E)$, $u \in U$. A similar situation occurs in the case of deformations of complex structure.

Suppose $v \in C^\infty(E)$ and let F_v^1 denote its time-1 flow defined by (2.9), so that in a splitting for E with curvature H, we have $v = X + \xi$ and by (2.10),

$$F_v^1 = \varphi_*^1 e^{B_1}, \quad B_1 = \int_0^1 \varphi_s^*(i_X H + d\xi)\; ds, \tag{5.3}$$

where φ_*^t is the flow of the vector field X. The Courant isomorphism F_v^1 acts on generalized complex structures, taking a given deformation L_ϵ to $F_v^1(L_\epsilon)$. If v has sufficiently small 1-jet, then $F_v^1(L_\epsilon)$ may be expressed as $L_{\epsilon'}$ for another section $\epsilon' \in C^\infty(\wedge^2 L^*)$, and we denote it $F_v^1(\epsilon) := \epsilon'$. We now determine an approximate formula for $F_v^1(\epsilon)$ in terms of (ϵ, v).

Proposition 5.4. *Let $\mathcal{J}$ be a generalized complex structure with $+i$-eigenbundle $L \subset E \otimes \mathbb{C}$, and let $\epsilon \in C^\infty(\wedge^2 L^*)$ be such that* (5.1) *is invertible. Then for $v \in C^\infty(E)$ with sufficiently small* 1*-jet, the time-*1 *flow* (5.3) *satisfies*

$$F_v^1(\epsilon) = \epsilon + d_L v^{0,1} + R(\epsilon, v), \tag{5.4}$$

where $v = v^{1,0} + v^{0,1}$ according to the splitting $L \oplus L^*$, and R satisfies

$$R(t\epsilon, tv) = t^2 \widetilde{R}(\epsilon, v, t),$$

where $\widetilde{R}(\epsilon, v, t)$ is smooth in t for small t.

Proof. Define $\epsilon(s,t)$ for $s, t \in \mathbb{R}$ by

$$\epsilon(s,t) = F_{tv}^1(s\epsilon), \tag{5.5}$$

so that $\epsilon = \epsilon(1,1)$ and $\epsilon(0) = 0$. We first compute the derivatives of (5.5) at $s = t = 0$. The derivative in s is easily computed:

$$\left.\frac{\partial \epsilon(s,t)}{\partial s}\right|_{(0,0)} = \frac{\partial (s\epsilon)}{\partial s} = \epsilon.$$

The derivative in t may be computed using the property of flows that $F_{tv}^1 = F_v^t$, together with the fact that the flow F_v^t is generated by the adjoint action $\mathrm{ad}(v) = [v, \cdot]$ of v on the Courant algebroid. Using the properties of the

Courant bracket, we obtain, for $y, z \in C^\infty(L)$,

$$\left.\frac{\partial \epsilon(s,t)}{\partial t}\right|_{(0,0)}(y,z) = \frac{\partial F_v^t(0)}{\partial t}(y,z) = \langle -[v,y], z\rangle = d_L v^{0,1}(y,z).$$

By Taylor's theorem we obtain

$$F_{tv}^1(s\epsilon) = s\epsilon + t d_L v^{0,1} + r(s,t,\epsilon,v),$$

where r is smooth of order $O(s^2, st, t^2)$ at zero. Setting $R(\epsilon, v) = r(1,1,\epsilon,v)$, we obtain the result, since clearly $r(1,1,t\epsilon,tv) = r(t,t,\epsilon,v)$ is of order $O(t^2)$. $\square$

Whereas the Maurer-Cartan equation (5.2) indicates that, infinitesimally, deformations of generalized complex structure lie in $\ker d_L \subset C^\infty(\wedge^2 L^*)$, the previous proposition shows us that, infinitesimally, deformations which differ by sections which lie in the image of d_L are equivalent. Hence we expect the tangent space to the moduli space to lie in the Lie algebroid cohomology $H^2(M, L)$, which by ellipticity is finite-dimensional for M compact. We now develop the Hodge theory required to prove this assertion.

We follow the usual treatment of Hodge theory as described in [38]. Choose a Hermitian metric on the complex Lie algebroid L and let $|\varphi|_k$ be the L_k^2 Sobolev norm on sections $\varphi \in C^\infty(\wedge^p L^*)$ induced by the metric. We then have the elliptic, self-adjoint Laplacian

$$\Delta_L = d_L d_L^* + d_L^* d_L.$$

Let $\mathcal{H}^p$ be the space of Δ_L-harmonic forms, which is isomorphic to $H^p(M, L)$ by the standard argument and let H be the orthogonal projection of $C^\infty(\wedge^p L)$ onto the closed subspace $\mathcal{H}^p$. Also, let G be the Green smoothing operator quasi-inverse to Δ_L, i.e. $G\Delta + H = \mathrm{Id}$ and

$$G : L_k^2 \to L_{k+2}^2.$$

We will find it useful, as Kuranishi did, to define the once-smoothing operator

$$Q = d_L^* G : L_k^2 \to L_{k+1}^2,$$

which then satisfies

$$\begin{aligned} &\mathrm{Id} = H + d_L Q + Q d_L, \\ &Q^2 = d_L^* Q = Q d_L^* = HQ = QH = 0. \end{aligned} \tag{5.6}$$

We now have the algebraic and analytical tools required to prove a direct analog of Kuranishi's theorem for generalized complex manifolds.

5.2. *The deformation theorem.*

THEOREM 5.5. *Let $(M, \mathcal{J})$ be a compact generalized complex manifold. There exists an open neighbourhood $U \subset H^2(M, L)$ containing zero, a smooth family $\widetilde{\mathcal{M}} = \{\epsilon(u) \ : \ u \in U, \ \epsilon(0) = 0\}$ of generalized almost complex deformations of $\mathcal{J}$, and an analytic obstruction map $\Phi : U \to H^3(M, L)$ with $\Phi(0) = 0$ and $d\Phi(0) = 0$, such that the deformations in the sub-family $\mathcal{M} = \{\epsilon(z) \ : \ z \in \mathcal{Z} = \Phi^{-1}(0)\}$ are precisely the integrable ones. Furthermore, any sufficiently small deformation ϵ of $\mathcal{J}$ is equivalent to at least one member of the family $\mathcal{M}$. In the case that the obstruction map vanishes, $\mathcal{M}$ is a smooth locally complete family.*

Proof. The proof is divided into two parts: first, we construct a smooth family $\widetilde{\mathcal{M}}$, and show it contains the family of integrable deformations $\mathcal{M}$ defined by the map Φ; second, we describe its miniversality property. We follow the paper of Kuranishi [25], where more details can be found.

Part I: For sufficiently large k, $L^2_k(M, \mathbb{R})$ is a Banach algebra (see [31]), and the map $f : \epsilon \mapsto \epsilon + \frac{1}{2}Q[\epsilon, \epsilon]$ extends to a smooth map

$$f : L^2_k(\wedge^2 L^*) \longrightarrow L^2_k(\wedge^2 L^*),$$

whose derivative at the origin is the identity mapping. By the inverse function theorem, f^{-1} maps a neighbourhood of the origin in $L^2_k(\wedge^2 L^*)$ smoothly and bijectively to another neighbourhood of the origin. Hence, for sufficiently small $\delta > 0$, the finite-dimensional subset of harmonic sections,

$$U = \{u \in \mathcal{H}^2 < L^2_k(\wedge^2 L^*) \ : \ |u|_k < \delta\},$$

defines a family of sections as follows:

$$\widetilde{M} = \{\epsilon(u) = f^{-1}(u) \ : \ u \in U\},$$

where $\epsilon(u)$ depends smoothly (in fact, holomorphically) on u, and satisfies $f(\epsilon(u)) = u$. Applying the Laplacian to this equation, we obtain

$$\Delta_L \epsilon(u) + \tfrac{1}{2} d_L^*[\epsilon(u), \epsilon(u)] = 0.$$

This is a quasi-linear elliptic PDE, and by a result of Morrey [29], we conclude that the solutions $\epsilon(u)$ of this equation are actually smooth, i.e.

$$\epsilon(u) \in C^\infty(\wedge^2 L^*).$$

Hence we have constructed a smooth family of generalized almost complex deformations of $\mathcal{J}$, over an open set $U \subset \mathcal{H}^2 \cong H^2(M, L)$.

We now ask which of these deformations satisfy the Maurer-Cartan equation (5.2). By definition of $\epsilon(u)$, and using (5.6), we obtain

$$\begin{aligned} d_L\epsilon(u) + \tfrac{1}{2}[\epsilon(u),\epsilon(u)] &= -\tfrac{1}{2}d_L Q[\epsilon(u),\epsilon(u)] + \tfrac{1}{2}[\epsilon(u),\epsilon(u)] \\ &= \tfrac{1}{2}(Qd_L + H)[\epsilon(u),\epsilon(u)]. \end{aligned}$$

Since the images of Q and H are L^2-orthogonal, we see that $\epsilon(u)$ is integrable if and only if $H[\epsilon(u),\epsilon(u)] = Qd_L[\epsilon(u),\epsilon(u)] = 0$. We now use the argument of Kuranishi [25] which, using the compatibility between $[\cdot,\cdot]$ and d_L, shows that $Qd_L[\epsilon(u),\epsilon(u)]$ vanishes if $H[\epsilon(u),\epsilon(u)]$ does.

Hence, $\epsilon(u)$ is integrable precisely when u lies in the vanishing set of the analytic mapping $\Phi : U \to H^3_L(M)$ defined by

$$\Phi(u) = H[\epsilon(u),\epsilon(u)]. \tag{5.7}$$

Note furthermore that $\Phi(0) = d\Phi(0) = 0$.

Part II: For the second part of the proof, we give an alternative characterisation of the family $\mathcal{M} = \{\epsilon(z)\ :\ z \in \mathcal{Z} = \Phi^{-1}(0)\}$. We claim that $\mathcal{M}$ is actually a neighbourhood around zero in the set

$$\mathcal{M}' = \left\{\epsilon \in C^\infty(\wedge^2 L^*)\ :\ d_L\epsilon + \tfrac{1}{2}[\epsilon,\epsilon] = 0,\quad d_L^*\epsilon = 0\right\}.$$

To show this, let $\epsilon(u) \in \mathcal{M}$. Then since $\epsilon(u) = u - \frac{1}{2}Q[\epsilon(u),\epsilon(u)]$ and $d_L^*Q = 0$, we see that $d_L^*\epsilon(u) = 0$, showing that $\mathcal{M} \subset \mathcal{M}'$. Conversely, let $\epsilon \in \mathcal{M}'$. Then since $d_L^*\epsilon = 0$, applying d_L^* to the equation $d_L\epsilon + \frac{1}{2}[\epsilon,\epsilon] = 0$ we obtain $\Delta_L\epsilon + \frac{1}{2}d_L^*[\epsilon,\epsilon] = 0$, and applying Green's operator we see that $\epsilon + \frac{1}{2}Q[\epsilon,\epsilon] = H\epsilon$, i.e. $F(\epsilon) = H\epsilon \in \mathcal{H}^2$, proving that a small open set in $\mathcal{M}'$ is contained in $\mathcal{M}$, completing the argument.

We now show that every sufficiently small deformation of the generalized complex structure is equivalent to one in our finite-dimensional family $\mathcal{M}$. Let $P \subset C^\infty(L^*)$ be the L^2 orthogonal complement of $\ker d_L \subset C^\infty(L^*)$, or in other words, sections in the image of d_L^*. We show that there exist neighbourhoods of the origin $V \subset C^\infty(\wedge^2 L^*)$ and $W \subset P$ such that for any $\epsilon \in V$ there is a unique $v \in C^\infty(E)$ such that $v^{0,1} \in W$ and the time-1 flow $F_v^1(\epsilon)$ satisfies

$$d_L^* F_v^1(\epsilon) = 0. \tag{5.8}$$

This would imply that any sufficiently small solution to $d_L\epsilon + \frac{1}{2}[\epsilon,\epsilon] = 0$ is equivalent to another solution ϵ' such that $d_L^*\epsilon' = 0$, i.e. a solution in $\mathcal{M}' = \mathcal{M}$. Extended to smooth families, this result would prove local completeness.

We see from (5.4) that (5.8) holds if and only if

$$d_L^*\epsilon + d_L^* d_L v^{0,1} + d_L^* R(\epsilon, v) = 0.$$

Assuming $v^{0,1} \in P$, we see that $d_L^* v^{0,1} = Hv^{0,1} = 0$, so that

$$d_L^*\epsilon + \Delta_L v^{0,1} + d_L^* R(\epsilon, v) = 0. \tag{5.9}$$

Applying the Green operator G, we obtain

$$v^{0,1} + Q\epsilon + QR(\epsilon, v) = 0. \tag{5.10}$$

By the definition (5.4) of $R(\epsilon, v)$, the map

$$F : (\epsilon, v^{0,1}) \mapsto v^{0,1} + Q\epsilon + QR(\epsilon, v)$$

is continuous from a neighbourhood of the origin $V_0 \times W_0$ in $C^\infty(\wedge^2 L^*) \times P$ to P, where all spaces are endowed with the L^2_k norm, k sufficiently large. Also, the derivative of F with respect to $v^{0,1}$ (at 0) is the identity map. Therefore by the implicit function theorem, there are neighbourhoods $V \subset V_0, W_1 \subset \widehat{W_0}$ such that given $\epsilon \in V$, equation (5.10), i.e. $F = 0$, is satisfied for a unique $v^{0,1} \in W_1$, and which depends smoothly on $\epsilon \in V$. Furthermore, since $\epsilon \in V$ is itself smooth, the unique solution v satisfies the quasi-linear elliptic PDE (5.9), implying that v is smooth as well, hence $v^{0,1}$ lies in the neighbourhood $W = W_1 \cap P$. Therefore we have shown that every sufficiently small deformation of the generalized complex structure is equivalent to one in our finite-dimensional family $\mathcal{M}$.

If the obstruction map Φ vanishes, so that $\mathcal{M}$ is a smooth family, then given any other smooth family $\mathcal{M}_S = \{\epsilon(s) \ : \ s \in S, \ \epsilon(s_0) = 0\}$ with basepoint $s_0 \in S$, the above argument provides, for s in some neighbourhood T of s_0, a smooth family of sections $v(s) \in C^\infty(E)$ whose time-1 flow takes each $\epsilon(s)$ to $\epsilon(f(s))$, $f(s) \in U \subset H^2(M, L)$. This defines a smooth map $f : T \to U$, $f(s_0) = 0$, such that $f^*\mathcal{M} = \mathcal{M}_S$. Thus we establish that $\mathcal{M}$ is a locally complete family of deformations. $\square$

5.3. *Examples of deformed structures.* Consider deforming a compact complex manifold (M, J) as a generalized complex manifold. Recall that the associated Lie algebroid is $L = T_{0,1} \oplus T^*_{1,0}$, so the deformation complex is simply the holomorphic multivector Dolbeault complex $\left(\Omega^{0,\bullet}(\wedge^\bullet T_{1,0}), \overline{\partial}\right)$. The base of the Kuranishi family therefore lies in the finite-dimensional vector space

$$H^2(M, L) = \oplus_{p+q=2} H^q(M, \wedge^p T_{1,0}),$$

whereas the image of the obstruction map lies in

$$H^3(M, L) = \oplus_{p+q=3} H^q(M, \wedge^p T_{1,0}).$$

In this way, generalized complex manifolds provide a geometrical interpretation of the "extended complex deformation space" defined by Kontsevich and Barannikov [4]. Any deformation ϵ has three components

$$\beta \in H^0(M, \wedge^2 T_{1,0}), \quad \varphi \in H^1(M, T_{1,0}), \quad B \in H^2(M, \mathcal{O}).$$

The component φ is a usual deformation of the complex structure, as described by Kodaira and Spencer. The component B represents a residual action by cohomologically nontrivial B-field transforms; these do not affect the type. The component β, however, is a new deformation for complex manifolds. Setting $B = \varphi = 0$, the integrability condition reduces to

$$\overline{\partial}\beta + \tfrac{1}{2}[\beta, \beta] = 0,$$

which is satisfied if and only if the bivector β is holomorphic and Poisson. Writing $\beta = \frac{-1}{4}(Q + iP)$ for Q, P real bivectors of type $(0, 2) + (2, 0)$ such that $Q = PJ^*$, we may explicitly determine the deformed generalized complex structure:

$$\mathcal{J}_\beta = e^{\beta+\overline{\beta}}\mathcal{J}_J e^{-(\beta+\overline{\beta})} = \begin{pmatrix} J & P \\ & -J^* \end{pmatrix}.$$

In this way, we obtain a new class of generalized complex manifolds with type controlled by the rank of the holomorphic Poisson bivector β.

Example 5.6 (Deformed generalized complex structure on $\mathbb{C}P^2$). For $\mathbb{C}P^2$, $\wedge^2 T_{1,0} = \mathcal{O}(3)$, and for dimensional reasons, any holomorphic bivector $\beta \in H^0(M, \mathcal{O}(3))$ is automatically Poisson. Hence any holomorphic section of $\mathcal{O}(3)$ defines an integrable deformation of the complex structure into a generalized complex structure. Since $H^1(T_{1,0}) = H^2(\mathcal{O}) = 0$, we may conclude from the arguments above that the locally complete family of deformations is smooth and of complex dimension 10. However, one can also check that the obstruction space vanishes in this case, by the Bott formulae.

The holomorphic Poisson structure β has maximal rank outside its vanishing locus, which must be a cubic curve C. Hence the deformed generalized complex structure is of B-symplectic type (type 0) outside C and of complex type (type 2) along the cubic. The complexified symplectic form $B + i\omega = \beta^{-1}$ is singular along C. We therefore have an example of a compact generalized complex manifold exhibiting type change along a codimension 2 subvariety.

Note that while holomorphic Poisson bivectors may be thought of as infinitesimal noncommutative deformations in the sense of quantization of Poisson structures, we are viewing them here as genuine (finite) deformations of the generalized complex structure. For more details about this distinction and its consequences, see [19], [16].

6. **Generalized complex branes**

In this section we introduce the natural "sub-objects" of generalized complex manifolds, generalizing both holomorphic submanifolds of a complex manifold and Lagrangian submanifolds in symplectic geometry. In fact, even in the case of a usual symplectic manifold, there are generalized complex branes besides the Lagrangian ones: we show these are the *coisotropic A-branes* discovered by Kapustin and Orlov [21].

As has been emphasized by physicists, a geometric description of a brane in M involves not only a submanifold $\iota : S \longrightarrow M$ but also a vector bundle supported on it; in cases where a nontrivial S^1-gerbe G is present one replaces the vector bundle by an *object* ("twisted vector bundle") of the pullback gerbe $\iota^* G$. Since the Courant bracket captures the differential geometry of a gerbe, we obtain a convenient description of branes in terms of generalized geometry. For simplicity we shall restrict our attention to branes supported on loci where the pullback gerbe $\iota^* G$ is trivializable. We begin by phrasing the definition of a gerbe trivialization in terms of the Courant bracket. Recall that $\iota^* E$ denotes the pullback of exact Courant algebroids, defined in the appendix.

Definition 6.1. Let E be an exact Courant algebroid on M and let $\iota : S \longrightarrow M$ be a submanifold. A (Courant) *trivialization* of E along S consists of a bracket-preserving isotropic splitting $s : TS \longrightarrow \iota^* E$ inducing an isomorphism

$$s + \pi^* : (TS \oplus T^*S, [\cdot,\cdot]_0) \longrightarrow \iota^* E.$$

If an isotropic splitting $\tilde{s} : TM \longrightarrow E$ is chosen, with curvature $H \in \Omega^3_{cl}(M)$, then $\iota^* E$ inherits a splitting with curvature $\iota^* H$, and any trivialization (ι, s) is characterized by the difference $s - \iota^* \tilde{s} = F \in \Omega^2(S)$, which satisfies

$$\iota^* H = dF. \tag{6.1}$$

Therefore the gerbe curvature is exact when pulled back to S. Indeed, we obtain a generalized pullback morphism $\rho \mapsto e^F \wedge \iota^* \rho$, defining a map from the twisted de Rham complex of M to the usual de Rham complex of S:

$$(\Omega^\bullet(M), d_H) \xrightarrow{e^F \iota^*} (\Omega^\bullet(S), d).$$

This may be viewed as the the image under the Chern character of a morphism from the twisted K-theory of M to the usual K-theory of S.

To avoid confusion, let $E|_S$ denote the restriction of the bundle E to S, as opposed to $\iota^* E = K^\perp / K$, for $K = \mathrm{Ann}(TS)$, which defines the pullback Courant algebroid over S. The trivialization (ι, s) defines a maximal isotropic subbundle $s(TS) \subset \iota^* E$. Further, the quotient map $q : K^\perp \longrightarrow K^\perp / K$ determines a bijection taking maximal isotropic subbundles $L \subset \iota^* E$ to maximal isotropic subbundles $q^{-1}(L) \subset E|_S$ contained in $K^\perp$.

Definition 6.2. The *generalized tangent bundle* to the trivialization $\mathcal{L} = (\iota, s)$ of E is the maximal isotropic subbundle $\tau_{\mathcal{L}} \subset E|_S$ defined by $\tau_{\mathcal{L}} = q^{-1}(s(TS))$.

Note that $\mathrm{Ann}(TS) = N^*S$, so that $\tau_{\mathcal{L}}$ is actually an extension of the tangent bundle by the conormal bundle:

$$0 \longrightarrow N^*S \longrightarrow \tau_{\mathcal{L}} \xrightarrow{\pi} TS \longrightarrow 0. \tag{6.2}$$

If a splitting $\tilde{s} : TM \longrightarrow E$ is chosen, with $s - \iota^*\tilde{s} = F \in \Omega^2(S)$ as in (6.1), then $\tau_{\mathcal{L}}$ has the explicit form

$$\tau_{\mathcal{L}} = \{X + \eta \in TS \oplus T^*M \ : \ \iota^*\eta = i_X F\}. \tag{6.3}$$

Comparing this with Proposition 2.7, we obtain the following canonical example of a Courant trivialization:

Example 6.3. Let $L \subset (TM \oplus T^*M, [\cdot,\cdot]_H)$ be a Dirac structure and let $\iota : S \hookrightarrow M$ be a maximal integral submanifold for the (generalized) distribution $\Delta = \pi(L) \subset TM$. Then along S, we have $L = L(\Delta, \varepsilon)$ for a unique $\varepsilon \in \Omega^2(S)$, and by the same argument as in Proposition 2.7, we obtain

$$\iota^* H = d\varepsilon.$$

Therefore we see that a Dirac structure induces a (generalized) foliation of the manifold by trivializations $\mathcal{L} = (\iota, \varepsilon)$.

Note that in this example, $\tau_{\mathcal{L}} = L|_S$ inherits a Lie algebroid structure over S, since any sections $u, v \in C^\infty(S, \tau_{\mathcal{L}})$ may be extended to $\tilde{u}, \tilde{v} \in C^\infty(M, L)$ and then the expression

$$[u, v] := [\tilde{u}, \tilde{v}]|_S$$

is independent of extension and defines a Lie bracket. For general trivializations, however, the ambient Dirac structure L is unavailable, and the argument fails.

A complex submanifold $S \subset M$ of a complex manifold is defined by the property that $J(TS) = TS$. Similarly, we define a compatibility condition between a Courant trivialization and a generalized complex structure.

Definition 6.4. A Courant trivialization $\mathcal{L} = (\iota, s)$ is said to be *compatible* with the generalized complex structure $\mathcal{J}$ if and only if

$$\mathcal{J}(\tau_{\mathcal{L}}) = \tau_{\mathcal{L}},$$

i.e. its generalized tangent bundle is a complex subbundle of E.

An immediate consequence of the definition is that $\pi(\mathcal{J}(N^*S)) \subset TS$, which by (3.17) is the statement that $P(N^*S) \subset TS$, i.e. S is a coisotropic submanifold for the Poisson structure P. Since P is Poisson, $\Delta = P(N^*S)$ integrates to a singular foliation called the *characteristic foliation* of S.

Decomposing $\tau_{\mathcal{L}} \otimes \mathbb{C}$ into $\pm i$-eigenspaces for $\mathcal{J}$, we obtain

$$\tau_{\mathcal{L}} \otimes \mathbb{C} = \ell \oplus \overline{\ell}.$$

Note that the isotropic subbundle $\ell \subset (E \otimes \mathbb{C})|_S$ is contained in the $+i$-eigenbundle L of $\mathcal{J}$, i.e.

$$\ell \subset L|_S.$$

Therefore, the argument of Example 6.3 concerning restriction of Courant brackets applies and we obtain the following result.[2]

PROPOSITION 6.5. *Let $\mathcal{J}$ be a generalized complex structure and let $\mathcal{L}$ be a compatible Courant trivialization. Define $\ell = \ker(\mathcal{J} - i) \cap (\tau_{\mathcal{L}} \otimes \mathbb{C})$. Then the Courant bracket induces a Lie bracket on $C^\infty(S, \ell)$, making $(\ell, [\cdot, \cdot], \pi)$ into a complex Lie algebroid over S.*

The associated Lie algebroid complex $(C^\infty(S, \wedge^\bullet \ell^*), d_\ell)$ is actually *elliptic*, by the same reasoning as in Proposition 3.12, and may be used to study the deformation theory of $\mathcal{L}$, which we leave for a future work.

The Lie algebroid ℓ projects to a generalized distribution $A = \pi(\ell) \subset TS \otimes \mathbb{C}$, which is integrable and satisfies $A + \overline{A} = TS \otimes \mathbb{C}$. The intersection $A \cap \overline{A} = \Delta \otimes \mathbb{C}$ coincides with the characteristic distribution of the coisotropic submanifold S. Therefore, by the reasoning in Proposition 4.2, wherever Δ has constant rank, A defines an invariant integrable holomorphic structure transverse to the characteristic foliation.

COROLLARY 6.6. *Let $\mathcal{L}$ be a compatible trivialization and let ℓ be the complex Lie algebroid defined above. In a neighbourhood where the characteristic distribution is of constant rank, $A = \pi(\ell) \subset TS \otimes \mathbb{C}$ defines an integrable holomorphic structure transverse to the characteristic foliation, which descends to the leaf space.*

Since ℓ is a Lie algebroid, we may associate to any compatible trivialization $\mathcal{L}$ the category of ℓ-modules, i.e. complex vector bundles V over S, equipped with flat Lie algebroid connections with respect to ℓ. We call these *generalized complex branes* on $\mathcal{L}$.

Definition 6.7 (Generalized complex brane). Let $\mathcal{J}$ be a generalized complex structure and $\mathcal{L}$ a compatible trivialization. A *generalized complex brane* supported on $\mathcal{L}$ is a module over the Lie algebroid ℓ.

Example 6.8 (Complex branes). Let $\mathcal{L} = (\iota, F)$, for $\iota : S \hookrightarrow M$ and $F \in \Omega^2(S)$, be a generalized complex trivialization in a complex manifold, so that $\iota^* H = dF$ and $\tau_{\mathcal{L}}$, given by (6.3), is a complex subbundle for the complex structure $\mathcal{J}_J$. This happens if and only if

[2]This Lie algebroid was obtained independently by Kapustin and Li [20], as defining the BRST complex describing open strings with both ends on $\mathcal{L}$.

- $TS \subset TM$ is a complex subbundle for J, i.e. S is a complex submanifold, and
- $J^* i_X F + i_{JX} F \in N^*S$ for all $X \in TS$, i.e. F is of type $(1,1)$.

In this case, the Lie algebroid ℓ is given by

$$\ell = \{X + \xi \in T_{0,1}S \oplus T^*_{1,0}M \ : \ \iota^*\xi = i_X F\}$$

and is therefore isomorphic to $T_{0,1}S \oplus N^*_{1,0}S$, where $N^*_{1,0}S$ denotes the holomorphic conormal bundle of S. As a result, a generalized complex brane supported on $\mathcal{L}$ consists of a holomorphic vector bundle V over S, together with a holomorphic section $\phi : V \longrightarrow N_{1,0}S \otimes V$ satisfying

$$\phi \wedge \phi = 0 \in H^0(S, \wedge^2 N_{1,0}S \otimes \mathrm{End}(V)).$$

Example 6.9 (Symplectic branes). As in the previous example, let $\mathcal{L} = (\iota, F)$ be a compatible trivialization, but for a symplectic structure $\mathcal{J}_\omega$. If $F = 0$, then $\tau_{\mathcal{L}} = TS \oplus N^*S$, and $\mathcal{J}(\tau_{\mathcal{L}}) = \tau_{\mathcal{L}}$ is simply the requirement that $\omega^{-1}(N^*S) \subset TS$ and $\omega(TS) \subset N^*S$, i.e. S is a Lagrangian submanifold. In this particular case, $\ell = \{X - i\omega(X) \ : \ X \in TS\}$, so that ℓ is isomorphic as a Lie algebroid to TS itself; hence ℓ-modules are simply flat vector bundles supported on S.

However, there are symplectic branes beyond the flat bundles over Lagrangians if we allow $F \neq 0$; in general, as we saw in Corollary 6.6, S must be coisotropic and the Lie algebroid ℓ determines a complex distribution $A = \pi(\ell)$ defining an invariant holomorphic structure transverse to the characteristic foliation of S. However for a symplectic trivialization we have explicitly $\ell = (\tau_{\mathcal{L}} \otimes \mathbb{C}) \cap \Gamma_{-i\omega}$, and hence

$$\ell = \{X - i\omega(X) \in (TS \oplus T^*M) \otimes \mathbb{C} \ : \ i_X(F + i\iota^*\omega) = 0\}. \tag{6.4}$$

Since $A = \pi(\ell)$ and $\Delta \otimes \mathbb{C} = A \cap \overline{A}$ defines the characteristic foliation, (6.4) implies that $F + i\iota^*\omega$ is basic with respect to the foliation and defines a closed, nondegenerate $(0,2)$-form on the leaf space. Hence the leaf space inherits a natural holomorphic symplectic structure. In this way we obtain precisely the structure of *coisotropic A-brane*, discovered by Kapustin and Orlov [21] in their search for objects of the Fukaya category beyond the well-known Lagrangian ones.

For such coisotropic trivializations, ℓ is isomorphic as a Lie algebroid to the distribution $A = \pi(\ell)$, and so branes are vector bundles equipped with flat partial A-connections. This implies that they are flat along the characteristic distribution, transversally holomorphic and invariant along the distribution. Holomorphic bundles on the leaf space would provide examples.

Example 6.10 (Space-filling symplectic brane). A special case of the preceding example is when the submanifold S coincides with M itself; then any brane over $\mathcal{L} = (\mathrm{id}, s)$ is said to be *space-filling*. In this case, $\mathcal{L}$ may be described by a closed 2-form $F \in \Omega^2(M)$ such that

$$\sigma = F + i\omega$$

defines a holomorphic symplectic structure on M, with complex structure given by $J = -\omega^{-1}F$.

If M supports such a space-filling brane, then any complex submanifold $\iota : S \hookrightarrow M$ which is also coisotropic with respect to σ (for example, a complex hypersurface) defines a compatible trivialization $\mathcal{L}' = (\iota, \iota^* F)$ in M, and we may produce examples of branes on $\mathcal{L}'$ by pullback. The holomorphic symplectic structure on its leaf space is also known as the holomorphic symplectic reduction of $\mathcal{L}'$.

Example 6.11 (General space-filling branes). The existence of a space-filling generalized complex brane places a strong constraint on the generalized complex structure. Indeed, the generalized tangent bundle $\tau_{\mathcal{L}}$ determines an integrable isotropic splitting of the Courant algebroid

$$E = T^* \oplus \tau_{\mathcal{L}},$$

so that the curvature H vanishes. If $\mathcal{J}$ is the generalized complex structure, the constraint $\mathcal{J}(\tau_{\mathcal{L}}) = \tau_{\mathcal{L}}$ implies that $\mathcal{J}$ must have upper triangular form in this splitting:

$$\mathcal{J} = \begin{pmatrix} -J & P \\ & J^* \end{pmatrix}.$$

Here we use the canonical identification $\tau_{\mathcal{L}} = TM$. Since $\mathcal{J}$ is upper triangular, J is an integrable complex structure, for which $T_{0,1} = \ell$. The real Poisson structure P is of type $(2,0) + (0,2)$ as can be seen from the fact $JP = PJ^*$, and the complex bivector $\beta = \frac{-1}{4}(Q + iP)$, for $Q = PJ^*$, is such that the $+i$-eigenbundle of $\mathcal{J}$ can be written as $L = T_{0,1} \oplus \Gamma_\beta$, where β is viewed as a map $T^*_{1,0} \longrightarrow T_{1,0}$. Courant integrability then requires that β be a holomorphic Poisson structure. Therefore we see that space-filling branes only exist when $\mathcal{J}$ is a holomorphic Poisson deformation of a complex manifold.

In fact, one can show using a combination of the above arguments, or as is done in [39] by developing a theory of brane reduction, that for an arbitrary generalized complex brane, the Poisson and holomorphic structures transverse to the characteristic foliation (when it is regular) are compatible, defining an invariant transverse holomorphic Poisson structure.

Interesting relations between the coisotropic branes discussed in this section and noncommutative geometry have appeared in [18], [2] in particular; for more on this connection as well as the relation between coisotropic branes and generalized Kähler geometry, see [16].

7. **Appendix**

PROPOSITION 7.1. *Let E be an exact Courant algebroid over M with Ševera class $[H]$ and suppose $\iota : S \hookrightarrow M$ is a submanifold. Then $\iota^* E := K^\perp/K$, for $K = \mathrm{Ann}(TS) \subset E|_S$, inherits the structure of an exact Courant algebroid over S with Ševera class $\iota^*[H]$.*

Proof. We first show that ι^*E inherits a bracket. Let $u, v \in C^\infty(S, K^\perp/K)$, and choose representatives $u', v' \in C^\infty(S, K^\perp)$. Extend these over M as sections $\tilde{u}, \tilde{v} \in C^\infty(M, E)$. We claim that $[\tilde{u}, \tilde{v}]|_S$ defines a section of ι^*E which is independent of the choices made.

Firstly we observe that $[\tilde{u}, \tilde{v}]|_S \in C^\infty(S, K^\perp)$, since $\pi[\tilde{u}, \tilde{v}] = [\pi\tilde{u}, \pi\tilde{v}]$ and if X, Y are vector fields tangent to S then $[X, Y]$ is also tangent to S.

Secondly we claim that $[\tilde{u}, \tilde{v}] + K$ is independent of the choices made: for $p, q \in C^\infty(E)$ with $p|_S, q|_S \in C^\infty(S, K)$, we have $[\tilde{u} + p, \tilde{v} + q] - [\tilde{u}, \tilde{v}] = [\tilde{u}, q] + [p, \tilde{v}] + [p, q]$. Given any $x \in C^\infty(E)$ with $\pi(x)|_S \in C^\infty(S, TS)$, we verify that $\langle x, [\tilde{u}, q]\rangle = \pi(\tilde{u})\langle x, q\rangle - \langle [\tilde{u}, x], q\rangle$ vanishes upon restriction to S, since $\langle x, q\rangle$ vanishes along S and $\pi([\tilde{u}, x])$ is tangent to S. Similarly for the other two terms. This shows that $[\tilde{u}, \tilde{v}] + K$ is independent of the choices made. The remainder of the Courant algebroid properties are easily verified. □

PROPOSITION 7.2. *Let $L \subset E$ be a Dirac structure and assume that*

$$\iota^* L := \frac{L \cap K^\perp + K}{K}$$

is a smooth subbundle of ι^*E. Then it is a Dirac structure.

Proof. We need only verify that the maximal isotropic subbundle ι^*L is involutive. Let $u, v \in C^\infty(S, (L \cap K^\perp + K)/K)$. By the definition of the Courant bracket in Proposition 7.1, we may choose representatives $u', v' \in C^\infty(S, L \cap K^\perp + K)$ for u, v and extend these as sections of E over M in any way. In a neighbourhood $U \subset S$ where $L \cap K^\perp$ has constant rank, write $u' = x' + p'$, $v' = y' + q'$, where $x', y' \in C^\infty(U, L \cap K^\perp)$ and $p', q' \in C^\infty(U, K)$. Then choose extensions $x, y \in C^\infty(V, L)$ for x', y' and $p, q \in C^\infty(V, E)$ for p', q', where V is an open set in M containing U. Then

$$[x + p, y + q] = [x, y] + [x, q] + [p, y] + [p, q].$$

Since $x, y \in C^\infty(V, L)$ and $\pi(x), \pi(y)$ are tangent to S, we have $[x, y]|_S \in C^\infty(U, L \cap K^\perp)$. Also, $[x, q]|_S \in C^\infty(U, K)$ since, for $z \in C^\infty(V, E)$ with $\pi(z)$ tangent to S, we have $\langle z, [x, q]\rangle = \pi(x)\langle z, q\rangle - \langle [x, z], q\rangle$, which vanishes along S since z and $[x, z]$ are both tangent to S. The same argument applies to show $[p, y]|_S, [p, q]|_S \in C^\infty(S, K)$. This proves that

$$[x + p, y + q]|_S \in C^\infty(S, L \cap K^\perp + K)$$

and hence that $\iota^* L$ is involutive in U. Since $L \cap K^\perp$ has locally constant rank on an open dense set in S, this argument shows that $\iota^* L$ is involutive, as required. $\square$

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