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Reduction of Courant algebroids and generalized complex structures

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Abstract

We present a theory of reduction for Courant algebroids as well as Dirac structures, generalized complex, and generalized Kähler structures which interpolates between holomorphic reduction of complex manifolds and symplectic reduction. The enhanced symmetry group of a Courant algebroid leads us to define *extended* actions and a generalized notion of moment map. Key examples of generalized Kähler reduced spaces include new explicit bi-Hermitian metrics on $\mathbb{C}P^2$.

Keywords: Reduction; Courant algebroid; Dirac structure; Generalized complex geometry; Bi-Hermitian structure

Contents

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1. Introduction

In the presence of a symmetry, a given geometrical structure may, under suitable conditions, pass to the quotient. Often, however, the quotient does not inherit the same type of geometry as the original space; it may be necessary to pass to a further *reduction* for this to occur. For example, a complex manifold M admitting a holomorphic S^1 action certainly does not induce a complex structure on M/S^1; rather, one considers the complexification of this action to a $\mathbb{C}^*$ action, whose quotient, under suitable conditions, inherits a complex structure. Similarly, the quotient of a symplectic manifold by a symplectic S^1 action is never symplectic; rather it is endowed with a natural Poisson structure, whose leaves are the symplectic reduced spaces one desires.

In this paper we consider the reduction of generalized geometrical structures such as Dirac structures and generalized complex structures. These are geometrical structures defined not on the tangent bundle of a manifold but on the sum $TM \oplus T^*M$ of the tangent and cotangent bundles (or, more generally, on an exact Courant algebroid). These structures interpolate between many of the classical geometries such as symplectic and Poisson geometry, the geometry of foliations, and complex geometry. As a result the quotient procedure described in this paper interpolates between the known methods of reduction in these cases.

The main conceptual advance required to understand the reduction of generalized geometries is the fact that one must extend the notion of action of a Lie group on a manifold. Traditional geometries are defined in terms of the Lie bracket of vector fields, whose symmetries are given precisely by diffeomorphisms. As a result, one considers reduction in the presence of a group homomorphism from a Lie group into the group of diffeomorphisms. The Courant bracket, on the other hand, has an enhanced symmetry group which is an abelian extension of a diffeomorphism group by the group of closed 2-forms. For this reason one must consider actions which may have components acting nontrivially on the Courant algebroid while leaving the underlying manifold fixed. To formalize this insight, we introduce the notion of a *Courant algebra*, and explain how it acts on a Courant algebroid in a way which extends the usual action of a Lie algebra by tangent vector fields.

A surprising benefit of this point of view is that the concept of *moment map* in symplectic geometry obtains a new interpretation as an object which controls the extended part of the action mentioned above, that is, the part of the action trivially represented in the diffeomorphism group.

In preparing this article, the authors drew from a wide variety of sources, all of which provided hints toward the proper framework for generalized reduction. First, the literature on holomorphic reduction of complex manifolds as well as the field of Hamiltonian reduction of symplectic manifolds in the style of Marsden–Weinstein [24]. Also, in the original work of Courant and Weinstein [5,6] where the Courant bracket is introduced, some preliminary remarks about quotients can be found; subsequent formulations of reduction of Dirac structures appear in [2,3,22]. Most influential, however, has been the work of physicists on the problem of finding gauged sigma models describing supersymmetric sigma models with isometries. The reason this is relevant is that the geometry of a general $N = (2, 2)$ supersymmetric sigma model is equivalent to generalized Kähler geometry [9], and so any insight into how to "gauge" or quotient such a model provides us with guidance for the geometrical reduction problem. Our sources for this material have been the work of Hull, Roček, de Wit, and Spence [14,15], Witten [32], and Figueroa-O'Farill and Stanciu [8]. More recently in the physics literature, the gauging conditions have been re-interpreted in terms of the Courant bracket [7], a point of view which we develop and expand upon in this paper as well. Finally, in recent work of Hitchin [12], a natural generalized Kähler structure on the moduli space of instantons on a generalized Kähler 4-manifold is constructed by a method which amounts to an infinite-dimensional generalized Kähler quotient.

The paper is organized as follows. In Section 2 we review the definition of Courant algebroid, describe its group of symmetries, and define the concept of extended action. This involves the definition of a Courant algebra, a particular kind of Lie 2-algebra. In this section we also define a moment map for an extended action. In Section 3 we describe how an extended action on an exact Courant algebroid gives rise to reduced spaces equipped with induced exact Courant algebroids. It turns out that, even if the original Courant algebroid has trivial 3-form curvature, its reduced spaces may have nontrivial curvature. In Section 4 we arrive at the reduction procedure for generalized geometries, introducing an operation which transports Dirac structures from a Courant algebroid to its reduced spaces. This operation generalizes both the operation of Dirac push-forward and pull-back outlined in [4]. In Section 5 we apply this procedure to reduce generalized complex structures and provide several examples, including some with interesting type change. Finally in Section 6 we study a way to transport a generalized Kähler structure to the reduced spaces. This is very much in the spirit of the usual Kähler reduction procedure (see [10,18]). Finally we present two examples of generalized Kähler reduction: we produce generalized Kähler structures on $\mathbb{C}P^2$ with type change, first along a triple line (an example of which has been found in [12] using a different method) and second, along three distinct lines in the plane. These examples are particularly significant since they provide explicit bi-Hermitian metrics on $\mathbb{C}P^2$.

Recently there has been a great deal of interest in porting the techniques of Hamiltonian reduction to the setting of generalized geometry. The authors are aware of four other groups who have worked independently on this topic: Lin and Tolman [20], Stïenon and Xu [27], Hu [13], and Vaisman [29].

2. Symmetries of the Courant bracket

In this section we introduce an extended notion of group action on a manifold preserving twisted Courant brackets. We start by recalling the definition and basic properties of Courant algebroids.

2.1. Courant algebroids

Courant algebroids were introduced in [21] in order to axiomatize the properties of the Courant bracket, an operation on sections of $TM \oplus T^*M$ extending the Lie bracket of vector fields.

A *Courant algebroid* over a manifold M is a vector bundle $E \to M$ equipped with a fibrewise nondegenerate symmetric bilinear form $\langle \cdot,\cdot \rangle$, a bilinear bracket $[\cdot,\cdot]$ on the smooth sections $\Gamma(E)$, and a bundle map $\pi : E \to TM$ called the *anchor*, which satisfy the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

(C1) $[e_1,[e_2,e_3]] = [[e_1,e_2],e_3] + [e_2,[e_1,e_3]]$,
(C2) $\pi([e_1,e_2]) = [\pi(e_1), \pi(e_2)]$,
(C3) $[e_1, fe_2] = f[e_1,e_2] + (\pi(e_1)f)e_2$,
(C4) $\pi(e_1)\langle e_2,e_3\rangle = \langle [e_1,e_2],e_3\rangle + \langle e_2,[e_1,e_3]\rangle$,
(C5) $[e_1,e_1] = \mathcal{D}\langle e_1,e_1\rangle$,

where $\mathcal{D} = \frac{1}{2}\pi^* \circ d : C^\infty(M) \to \Gamma(E)$ (using $\langle\cdot,\cdot\rangle$ to identify E with E^*).

We see from axiom (C5) that the bracket is not skew-symmetric, but rather satisfies

$$[e_1,e_2] = -[e_2,e_1] + 2\mathcal{D}\langle e_1,e_2\rangle.$$

Since the left adjoint action is a derivation of the bracket (axiom (C1)), the pair $(\Gamma(E),[\cdot,\cdot])$ is a *Leibniz algebra* [23]. Note that the skew-symmetrization of this bracket does not satisfy the Jacobi identity; as was shown in [25], a Courant algebroid provides an example of an L_∞-algebra. We now briefly describe Ševera's classification of exact Courant algebroids.

Definition 2.1. A Courant algebroid is *exact* if the following sequence is exact:

$$0 \to T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \to 0. \tag{1}$$

Given an exact Courant algebroid, we may always choose a right splitting $\nabla : TM \to E$ which is *isotropic*, i.e. whose image in E is isotropic with respect to $\langle\cdot,\cdot\rangle$. Such a splitting has a curvature 3-form $H \in \Omega^3_{\mathrm{cl}}(M)$ defined as follows: for $X, Y \in \Gamma(TM)$,

$$i_Y i_X H = 2s\big[\nabla(X), \nabla(Y)\big], \tag{2}$$

where $s : E \to T^*M$ is the induced left splitting. Using the bundle isomorphism $\nabla + \frac{1}{2}\pi^* : TM \oplus T^*M \to E$, we transport the Courant algebroid structure onto $TM \oplus T^*M$. Given $X + \xi, Y + \eta \in \Gamma(TM \oplus TM^*)$, we obtain for the bilinear pairing

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}\big(\eta(X) + \xi(Y)\big), \tag{3}$$

and the bracket becomes

$$[X+\xi, Y+\eta]_H = [X,Y] + \mathcal{L}_X\eta - i_Y d\xi + i_Y i_X H, \tag{4}$$

which is the *H-twisted Courant bracket on* $TM \oplus T^*M$ [26]. Isotropic splittings of (1) differ by 2-forms $b \in \Omega^2(M)$, and a change of splitting modifies the curvature H by the exact form db. Hence the cohomology class $[H] \in H^3(M, \mathbb{R})$, called the *Ševera class*, is independent of the splitting and determines the exact Courant algebroid structure on E completely. When this class is integral, the exact Courant algebroid may be viewed as a generalized Atiyah sequence associated to a connection on an S^1 gerbe. In this sense, exact Courant algebroids arise naturally from the study of gerbes.

We now determine the symmetry group of an exact Courant algebroid, that is, the group of bundle automorphisms preserving the Courant algebroid structure.

Definition 2.2. The automorphism group $\mathrm{Aut}(E)$ of a Courant algebroid E is the group of bundle automorphisms $F : E \to E$ covering diffeomorphisms $\varphi : M \to M$ such that

(i) $\varphi^*\langle F\cdot, F\cdot\rangle = \langle\cdot,\cdot\rangle$, i.e. F is orthogonal,
(ii) $[F\cdot, F\cdot] = F[\cdot,\cdot]$, i.e. F is bracket-preserving,
(iii) $\pi \circ F = \varphi_* \circ \pi$, i.e. F is compatible with the anchor.

One can easily verify, using axiom (C3), that compatibility with the anchor is implied by requirements (i) and (ii) (see [9,16]).

Similarly, the Lie algebra of derivations $\mathrm{Der}(E)$ is the Lie algebra of linear first order differential operators D_X on $\Gamma(E)$, covering vector fields $X \in \Gamma(TM)$ such that $X\langle\cdot,\cdot\rangle = \langle D_X\cdot, \cdot\rangle + \langle\cdot, D_X\cdot\rangle$ and $D_X[\cdot,\cdot] = [D_X\cdot, \cdot] + [\cdot, D_X\cdot]$.

In the case of an exact Courant algebroid, one may choose an isotropic splitting of the anchor, inducing an isomorphism $E \cong TM \oplus T^*M$ as above, with bilinear pairing given by (3) and bracket given by (4). Now suppose that $F \in \mathrm{Aut}(E)$ covers $\varphi \in \mathrm{Diff}(M)$. Note that φ lifts naturally to $\Phi = \varphi_* + (\varphi^*)^{-1} \in \mathrm{End}(TM \oplus T^*M)$, which satisfies

$$[\Phi\cdot, \Phi\cdot]_H = \Phi[\cdot,\cdot]_{\varphi^* H}.$$

Therefore $\Phi^{-1}F$ is a fiber-preserving orthogonal map on $TM \oplus T^*M$ compatible with the anchor, which implies that it must be the orthogonal action of a 2-form $B \in \Omega^2(M)$ via $e^B : X + \xi \mapsto X + \xi + i_X B$ [9]. Since these "gauge transformations" satisfy

$$\left[e^B\cdot, e^B\cdot\right]_H = e^B[\cdot,\cdot]_{H+dB},$$

we see that $F = \Phi e^B$ is an automorphism if and only if $H - \varphi^* H = dB$. Therefore, the automorphism group consists of ordered pairs $(\varphi, B) \in \mathrm{Diff}(M) \times \Omega^2(M)$ such that $H - \varphi^* H = dB$, giving rise to the following splitting-independent description.

Proposition 2.3. *The automorphism group of an exact Courant algebroid E is an extension of the diffeomorphisms preserving the cohomology class $[H]$ by the abelian group of closed 2-forms*:

$$0 \to \Omega^2_{\mathrm{cl}}(M) \to \mathrm{Aut}(E) \to \mathrm{Diff}_{[H]}(M) \to 0. \tag{5}$$

If M is compact, the extension class in group cohomology is represented by the cocycle

$$c(\varphi_1, \varphi_2) = \varphi_1^{*-1}\left(Q - \varphi_2^{*-1} Q \varphi_2^*\right)\left(H - \varphi_1^* H\right),$$

where $Q = d^*G$, and d^*, G are the codifferential and Green operator with respect to a Riemannian metric.

Proof. Given an isotropic splitting of E with curvature H, and a Riemannian metric on the compact manifold M, we split the sequence (5) via the map $s : \mathrm{Diff}_{[H]}(M) \to \mathrm{Aut}(E)$ given by $s(\varphi) = (\varphi, B_\varphi)$, where $B_\varphi = Q(H - \varphi^* H)$. One can easily verify that $dB_\varphi = H - \varphi^* H$ and that

$$s(\varphi_1)s(\varphi_2)\bigl(s(\varphi_1\varphi_2)\bigr)^{-1} = \bigl(1, c(\varphi_1, \varphi_2)\bigr),$$

yielding the extension class. □

Differentiating a 1-parameter family of automorphisms $F_t = \Phi_t e^{tB}$, $F_0 = \mathrm{Id}$, we see that the Lie algebra $\mathrm{Der}(E)$ for a split exact Courant algebroid consists of pairs $(X, B) \in \Gamma(TM) \oplus \Omega^2(M)$ such that $\mathcal{L}_X H = dB$, which act via

$$(X, B) \cdot (Y + \eta) = \mathcal{L}_X(Y + \eta) + i_Y B. \tag{6}$$

We then have the following invariant description of derivations.

Proposition 2.4. *The Lie algebra of infinitesimal symmetries of an exact Courant algebroid E is an abelian extension of the Lie algebra of smooth vector fields by the closed 2-forms*:

$$0 \to \Omega^2_{\mathrm{cl}}(M) \to \mathrm{Der}(E) \to \Gamma(TM) \to 0. \tag{7}$$

The extension class in Lie algebra cohomology is represented by the cocycle

$$c(X, Y) = d i_X i_Y H.$$

Proof. Given an isotropic splitting of E with curvature H, we split sequence (7) via the map $s : \Gamma(TM) \to \mathrm{Der}(E)$ given by $s(X) = (X, i_X H)$. Then, using (6), we find that

$$\bigl[s(X), s(Y)\bigr] - s\bigl([X, Y]\bigr) = \bigl(0, c(X, Y)\bigr),$$

as required. □

This Lie algebra cocycle was also obtained by Hu in [13], where more details can be found.

It is immediately clear from axioms (C1), (C4) that $\Gamma(E)$ acts on itself by derivations via the left adjoint action $\mathrm{ad}_v(w) := [v, w]$. Unlike, however, the usual adjoint action of vector fields on the tangent bundle, the map $\mathrm{ad} : \Gamma(E) \to \mathrm{Der}(E)$ is neither surjective nor injective, as we now verify for exact Courant algebroids.

Proposition 2.5. *Let E be an exact Courant algebroid. Then the adjoint action* $\mathrm{ad} : v \mapsto [v, \cdot]$ *induces the following exact sequence*:

$$0 \to \Omega^1_{\mathrm{cl}}(M) \xrightarrow{\pi^*} \Gamma(E) \xrightarrow{\mathrm{ad}} \mathrm{Der}(E) \xrightarrow{\chi} H^2(M, \mathbb{R}) \to 0.$$

Proof. Given an isotropic splitting, we see from (4) that the kernel of the adjoint action is the space of closed 1-forms. Given any derivation $D_X = (X, B)$, we define $\chi(D_X) = [i_X H - B] \in H^2(M, \mathbb{R})$, which is surjective by the freedom to choose B, and whose kernel consists of (X, B) such that $B = i_X H - d\xi$, i.e. such that $(X, B) \cdot (Y + \eta) = [X + \xi, Y + \eta]$, proving exactness. □

2.2. Extended actions

Let a Lie group G act on a manifold M, so that we have the Lie algebra homomorphism $\psi : \mathfrak{g} \to \Gamma(TM)$. We wish to extend this action to a Courant algebroid E, making E into a G-equivariant vector bundle, in such a way that the Courant algebroid structure is preserved. In this section we show how this can be done by choosing an extension of $\mathfrak{g}$ equipped with a *Courant algebra* structure, and choosing a homomorphism from this extension to the Courant algebroid E.

Definition 2.6. A *Courant algebra* over the Lie algebra $\mathfrak{g}$ is a vector space $\mathfrak{a}$ equipped with a bilinear bracket $[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ and a map $\pi : \mathfrak{a} \to \mathfrak{g}$, which satisfy the following conditions for all $a_1, a_2, a_3 \in \mathfrak{a}$:

(c1) $[a_1, [a_2, a_3]] = [[a_1, a_2], a_3] + [a_2, [a_1, a_3]]$,
(c2) $\pi([a_1, a_2]) = [\pi(a_1), \pi(a_2)]$.

In other words, $\mathfrak{a}$ is a Leibniz algebra with a homomorphism to $\mathfrak{g}$.

A Courant algebroid provides an example of a Courant algebra over $\mathfrak{g} = \Gamma(TM)$, taking $\mathfrak{a} = \Gamma(E)$. Using the argument of Roytenberg–Weinstein [25], one sees that any Courant algebra is actually an example of a 2-term L_∞-algebra [1,19].

Definition 2.7. An *exact* Courant algebra is one for which π is surjective and $\mathfrak{h} = \ker \pi$ is abelian, i.e. $[h_1, h_2] = 0$ for all $h_1, h_2 \in \mathfrak{h}$.

For an exact Courant algebra, one obtains immediately an action of $\mathfrak{g}$ on $\mathfrak{h}$: $g \in \mathfrak{g}$ acts on $h \in \mathfrak{h}$ via $g \cdot h = [a, h]$, for any a such that $\pi(a) = g$. This is well defined since $\mathfrak{h}$ is abelian, and determines an action by axiom (c1). In fact there is a natural exact Courant algebra associated with any $\mathfrak{g}$-module, as we now explain.

Example 2.8 *(Hemisemidirect product).* Let $\mathfrak{g}$ be a Lie algebra acting on the vector space $\mathfrak{h}$. Then $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$ becomes a Courant algebra over $\mathfrak{g}$ via the bracket

$$\big[(g_1, h_1), (g_2, h_2)\big] = \big([g_1, g_2], g_1 \cdot h_2\big), \tag{8}$$

where $g \cdot h$ denotes the $\mathfrak{g}$-action. This bracket appeared in [17], where it was called the *hemisemidirect* product of $\mathfrak{g}$ with $\mathfrak{h}$. Note that in [31], Weinstein studied the case where $\mathfrak{g} = \mathfrak{gl}(V)$ and $\mathfrak{h} = V$, and called it an *omni-Lie algebra* due to the fact that, when $\dim V = n$, any n-dimensional Lie algebra can be embedded in it as a subalgebra.

Definition 2.9 *(Extended action).* Let G be a connected Lie group acting on a manifold M with infinitesimal action $\psi : \mathfrak{g} \to \Gamma(TM)$. An *extension* of this action to a Courant algebroid E over M is an exact Courant algebra $\mathfrak{a}$ over $\mathfrak{g}$ together with a Courant algebra morphism $\rho : \mathfrak{a} \to \Gamma(E)$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{a} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \psi} & & \\ & & & & \Gamma(E) & \longrightarrow & \Gamma(TM) & & \end{array}$$

which is such that $\mathfrak{h}$ acts trivially, i.e. $(\mathrm{ad} \circ \rho)(\mathfrak{h}) = 0$, and the induced action of $\mathfrak{g} = \mathfrak{a}/\mathfrak{h}$ on $\Gamma(E)$ integrates to a G-action on the total space of E.

The space of G-equivariant functions $f : M \to \mathfrak{g}^*$ acts on the space of extensions of an action ψ by the Courant algebra $\mathfrak{a}$, via $\rho(\cdot) \mapsto \rho(\cdot) + \mathcal{D}\langle f, \pi(\cdot)\rangle$, where $\mathcal{D} = \frac{1}{2}\pi^* \circ d$, as above; this generates an equivalence relation among extensions of actions.

Definition 2.10. Extensions ρ, ρ' of a given G-action to a Courant algebroid E, for a fixed Courant algebra $\mathfrak{a}$, are said to be *equivalent* if they agree upon restriction to $\mathfrak{h}$ and differ by a G-equivariant function $f : M \to \mathfrak{g}^*$, i.e.

$$\rho'(a) - \rho(a) = \mathcal{D}\big\langle f, \pi(a)\big\rangle.$$

Suppose now that the Courant algebroid in question is exact, as it will be in many cases of interest. Then an extended action is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{a} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & \Gamma(T^*M) & \longrightarrow & \Gamma(E) & \longrightarrow & \Gamma(TM) & \longrightarrow & 0 \end{array}$$

such that $\mathfrak{h}$ acts trivially, which occurs precisely when it acts via *closed* 1-forms, i.e. $\nu(\mathfrak{h}) \subset \Omega^1_{\mathrm{cl}}(M)$. Furthermore the induced $\mathfrak{g}$-action on E must integrate to a G-action (a priori, one has only the action of the universal cover of G). In order to make this condition more concrete, we observe that since we already know that the $\mathfrak{g}$-action on TM integrates to a G-action, one needs only to find a $\mathfrak{g}$-invariant splitting of E to guarantee that it is a G-bundle, as the splitting $E = TM \oplus T^*M$ carries a canonical G-equivariant structure.

Proposition 2.11. *Let the Lie group G act on the manifold M, and let $\mathfrak{a} \xrightarrow{\pi} \mathfrak{g}$ be an exact Courant algebra with a morphism ρ to an exact Courant algebroid E over M such that $\nu(\mathfrak{h}) \subset \Omega^1_{\mathrm{cl}}(M)$.*

If E has a $\mathfrak{g}$-invariant splitting, then the $\mathfrak{g}$-action on E integrates to an action of G, and hence ρ is an extended action of G on E. Conversely, if G is compact and ρ is an extended action, then by averaging splittings one can always find a $\mathfrak{g}$-invariant splitting of E.

The condition that a splitting is $\mathfrak{g}$-invariant can be expressed more concretely as follows. As shown in Section 2.1, a split exact Courant algebroid is isomorphic to the direct sum $TM \oplus T^*M$, equipped with the H-twisted Courant bracket for a closed 3-form H. In this splitting, therefore, for each $a \in \mathfrak{a}$ the section $\rho(a)$ decomposes as $\rho(a) = X_a + \xi_a$, and it acts via $[X_a + \xi_a, Y + \eta] = [X_a, Y] + \mathcal{L}_{X_a}\eta - i_Y d\xi_a + i_Y i_{X_a} H$, or as a matrix,

$$ad_{\rho(a)} = \begin{pmatrix} \mathcal{L}_{X_a} & 0 \\ i_{X_a}H - d\xi_a & \mathcal{L}_{X_a} \end{pmatrix}.$$

We see immediately from this that the splitting is preserved by this action if and only if for each $a \in \mathfrak{a}$,

$$i_{X_a}H - d\xi_a = 0. \tag{9}$$

We now provide a complete description, assuming G to be compact and E exact, of the simplest kind of extended action, namely one for which $\mathfrak{a} = \mathfrak{g}$.

Definition 2.12. A *trivially* extended G-action is one for which $\mathfrak{a} = \mathfrak{g}$ and $\pi : \mathfrak{a} \to \mathfrak{g}$ is the identity map, as described by the commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{id}} & \mathfrak{g} \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\psi} \\ \Gamma(E) & \longrightarrow & \Gamma(TM). \end{array}$$

Suppose that G is compact and E exact. By Proposition 2.11, we can always find a $\mathfrak{g}$-invariant splitting of E, so finding a trivially extended action ρ is equivalent to finding 1-forms ξ_a such that $\rho : a \mapsto X_a + \xi_a$ is a Courant algebra homomorphism (here $X_a = \psi(a)$ and $a \in \mathfrak{g}$). Preserving the bracket yields

$$\xi_{[a,b]} = \mathcal{L}_{X_a}\xi_b - i_{X_b}d\xi_a + i_{X_b}i_{X_a}H = \mathcal{L}_{X_a}\xi_b, \tag{10}$$

where (9) was used in the final equality. Equation (10) states that ξ_a is an equivariant form, and condition (9) can be phrased in terms of the Cartan model for G-equivariant cohomology. Recall that the Cartan complex of equivariant forms is the algebra of equivariant polynomial functions $\Phi : \mathfrak{g} \to \Omega^\bullet(M)$:

$$\Omega_G^k(M) = \bigoplus_{2p+q=k} \big(S^p\mathfrak{g}^* \otimes \Omega^q(M)\big)^G,$$

and the equivariant derivative d_G is defined by

$$(d_G\Phi)(a) = d\big(\Phi(a)\big) - i_{X_a}\Phi(a) \quad \forall a \in \mathfrak{g}.$$

Now consider the form $\Phi(a) = H + \xi_a$. Since the splitting is G-invariant, we have $\mathcal{L}_{X_a}H = 0$. Therefore Φ is an equivariant 3-form in the Cartan complex. Computing $d_G\Phi$, we obtain

$$d_G\Phi(a) = -\langle X_a + \xi_a, X_a + \xi_a\rangle = -\big\langle\rho(a), \rho(a)\big\rangle.$$

This shows that the quadratic form $c(a) = \langle\rho(a), \rho(a)\rangle$ is, firstly, constant along M, but also that it defines an invariant quadratic form on the Lie algebra $\mathfrak{g}$, and furthermore, one which is exact in the G-equivariant cohomology of M. If $c \equiv 0$, i.e. if the action is *isotropic*, then we see that the existence of a trivially extended G-action on E is determined by the equivariant extension of $[H]$. This last condition is well known to physicists in the context of gauging sigma models with Wess–Zumino term [14].

Theorem 2.13. *Let G be a compact Lie group. Then trivially extended G-actions on a fixed exact Courant algebroid with prescribed quadratic form $c(a) = \langle\rho(a), \rho(a)\rangle$ are, up to equivalence, in bijection with solutions to $d_G\Phi = c$ modulo d_G-exact forms, where $\Phi(a) = H + \xi_a$ is an equivariant 3-form and $[H] \in H^3(M, \mathbb{R})$ is the Ševera class of the Courant algebroid.*

Proof. The equivariant 3-form $\Phi(a) = H + \xi_a$ representing a trivially extended G-action ρ depends on a choice of $\mathfrak{g}$-invariant splitting for E. Changing the splitting by a gauge transformation $b \in \Omega^2(M)$, $\mathcal{L}_{X_a} b = 0$, the 3-form changes to $\Phi(a) = H + db + \xi_a + i_{X_a} b$. Also, an equivalent extended action ρ' satisfies $\rho'(a) - \rho(a) = df_a$ for a G-equivariant function f. The resulting equivariant 3-form is

$$\Phi'(a) = H + \xi_a + (db + i_{X_a} b + df_a).$$

But this is precisely the addition to Φ of an equivariantly exact 3-form, i.e. $\Phi' = \Phi + d_G\beta$, $\beta(a) = b + f_a$, proving the result. □

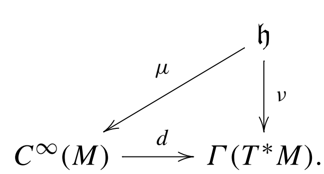
2.3. Moment maps for extended actions

Suppose that we have an extended G-action on an exact Courant algebroid as in the previous section, so that we have the map $\nu : \mathfrak{h} \to \Omega^1_{\mathrm{cl}}(M)$. Because the action is a Courant algebra morphism, this map is $\mathfrak{g}$-equivariant in the sense

$$\nu(g \cdot h) = \mathcal{L}_{\psi(g)} \nu(h). \tag{11}$$

Therefore we are led naturally to the definition of a moment map for this extended action, as an equivariant factorization of ν through the smooth functions.

Definition 2.14. A *moment map* for an extended $\mathfrak{g}$-action on an exact Courant algebroid is a $\mathfrak{g}$-equivariant map $\mu : \mathfrak{h} \to C^\infty(M, \mathbb{R})$ satisfying $d\mu = \nu$, i.e. such that the following diagram commutes:



Note that μ may be alternatively viewed as an equivariant map $\mu : M \to \mathfrak{h}^*$.

A moment map can be found only if two obstructions vanish. The first one is the induced map to cohomology $\nu_*:\mathfrak{h}\to H^1(M,\mathbb{R})$. Since (11) implies that ν_* always vanishes on $\mathfrak{g}\cdot\mathfrak{h}\subset\mathfrak{h}$, the first obstruction may be defined as an element

$$o_1\in H^0\big(\mathfrak{g},\mathfrak{h}^*\big)\otimes H^1(M,\mathbb{R}),$$

where the first term denotes Lie algebra cohomology with values in the module $\mathfrak{h}^*$. When this obstruction vanishes we may choose a lift $\tilde{\mu}:\mathfrak{h}\to C^\infty(M)$. The second obstruction results from the failure of this lift to be equivariant: consider the quantity $c(g,h)=\tilde{\mu}(g\cdot h)-\mathcal{L}_{\psi(g)}\tilde{\mu}(h)$ for $g\in\mathfrak{g},\ h\in\mathfrak{h}$. From (11) we conclude that c is a constant function along M. It is easily shown that this discrepancy, modulo changes of lift, defines an obstruction class

$$o_2\in H^1\big(\mathfrak{g},\mathfrak{h}^*\big).$$

Proposition 2.15. *A moment map for an extended $\mathfrak{g}$-action exists if and only if the obstructions $o_1\in H^0(\mathfrak{g},\mathfrak{h}^*)\otimes H^1(M,\mathbb{R})$ and $o_2\in H^1(\mathfrak{g},\mathfrak{h}^*)$ vanish. When it exists, a moment map is unique up to the addition of an element $\lambda\in\mathrm{Ann}(\mathfrak{g}\cdot\mathfrak{h})\subset\mathfrak{h}^*$.*

We now show how the usual notions of symplectic and Hamiltonian actions fit into the framework of extended actions of Courant algebras.

Example 2.16 *(Symplectic actions).* Let G be a Lie group acting on a symplectic manifold (M,ω) preserving the symplectic form, and let $\psi:\mathfrak{g}\to\Gamma(TM)$ denote the infinitesimal action. We now show that there is a natural extended action of the hemisemidirect product Courant algebra $\mathfrak{g}\oplus\mathfrak{g}$ on the standard Courant algebroid $TM\oplus T^*M$ with $H=0$. As described in Example 2.8, the Courant algebra is described by the sequence

$$0\to\mathfrak{g}\to\mathfrak{g}\oplus\mathfrak{g}\xrightarrow{\pi}\mathfrak{g}\to 0$$

and is equipped with the bracket

$$\big[(g_1,h_1),(g_2,h_2)\big]=\big([g_1,g_2],[g_1,h_2]\big). \tag{12}$$

Then define the action $\rho:\mathfrak{g}\oplus\mathfrak{g}\to\Gamma(TM\oplus T^*M)$ by

$$\rho(g,h)=X_g+i_{X_h}\omega,$$

where $X_g=\psi(g)$, for $g\in\mathfrak{g}$, and ω is the symplectic form. Then since

$$[X_{g_1}+i_{X_{h_1}}\omega,X_{g_2}+i_{X_{h_2}}\omega]=[X_{g_1},X_{g_2}]+\mathcal{L}_{X_{g_1}}i_{X_{h_2}}\omega=X_{[g_1,g_2]}+i_{X_{[g_1,h_2]}}\omega,$$

we see that ρ is a Courant morphism.

The question of finding a moment map for this extended action then becomes one of finding an equivariant map $\mu:\mathfrak{g}\to C^\infty(M)$ such that

$$d(\mu_g)=i_{X_g}\omega.$$

Hence we recover the usual moment map for a Hamiltonian action on a symplectic manifold.

Note that in this formalism, the notion of moment map is no longer tied to the geometry, i.e. the symplectic form. Instead, it is a constituent of the extended action. In fact, given an equivariant map $\mu : M \to \mathfrak{h}^*$ for a $\mathfrak{g}$-module $\mathfrak{h}$, one can naturally construct an extended action for which μ is a moment map, as we now indicate.

Proposition 2.17. *Given a $\mathfrak{g}$-equivariant map $\mu : M \to \mathfrak{h}^*$, where M is a G-space and $\mathfrak{h}$ a $\mathfrak{g}$-module, there is an induced extended action of the Courant algebra $\mathfrak{g} \oplus \mathfrak{h}$ with bracket* (8) *on the exact Courant algebroid $TM \oplus T^*M$ with $H = 0$, given by*

$$\rho : (g, h) \mapsto X_g + d(\mu_h),$$

where as before $X_g = \psi(g)$ is the infinitesimal $\mathfrak{g}$-action.

More generally, given a trivially extended action $\rho : \mathfrak{g} \to \Gamma(E)$ on an exact Courant algebroid, it can be extended to an action of $\mathfrak{g} \oplus \mathfrak{h}$ as above by any equivariant map $\mu : M \to \mathfrak{h}^*$ via the same formula

$$\tilde{\rho} : (g, h) \mapsto \rho(g) + d(\mu_h).$$

3. Reduction of Courant algebroids

In this section we develop a reduction procedure for exact Courant algebroids which can be seen as an "odd" analog of the usual notion of symplectic reduction due to Marsden and Weinstein [24]. A key observation is that an extended G-action on an exact Courant algebroid E over a manifold M does *not* necessarily induce an exact Courant algebroid on M/G, but rather one may need to pass to a suitably chosen submanifold $P \subset M$, in such a way that the *reduced space* $P/G = M_{\mathrm{red}}$ obtains an exact Courant algebroid. This is directly analogous to the well-known fact that, for a symplectic G-space M, the reduced spaces are the leaves of the Poisson structure inherited by M/G.

3.1. Reduction procedure

As we saw in the previous section, an extended action of a connected Lie group G on a Courant algebroid E over M makes E into an equivariant G-bundle in such a way that the Courant structure is preserved by the G-action. Therefore, assuming the G-action on the base to be free and proper, we obtain a Courant algebroid E/G over M/G. However, even if E were exact, E/G would certainly *not* be an exact Courant algebroid, since its rank is too large. We will see in this section how this construction can be modified so as to yield an *exact* reduced Courant algebroid.

So let G be a connected Lie group and E be an *exact* Courant algebroid over M. The first basic observation is that an extended action $\rho : \mathfrak{a} \times M \to E$ determines two natural distributions in E: the image of ρ, $K = \rho(\mathfrak{a})$, and its orthogonal, $K^\perp$. Recall that the action of $g \in \mathfrak{g}$ on any generating section $\rho(a)$ of K is simply

$$g \cdot \rho(a) = \big[\rho(\tilde{g}), \rho(a)\big] = \rho\big([\tilde{g}, a]\big) \in \Gamma(K),$$

where $\tilde{g} \in \mathfrak{a}$ is any lift of g, $\pi(\tilde{g}) = g$. It follows that K is a G-invariant distribution and, since the G-action on E preserves the symmetric pairing $\langle \cdot, \cdot \rangle$, $K^\perp$ is a G-invariant distribution as well.

It is clear that if ρ has constant rank, then the distributions K and $K^{\perp}$ are subbundles of E, but we do not make this global assumption at this point.

Definition 3.1. Given an extended action with image distribution $\rho(\mathfrak{a}) = K \subset E$, define the *big distribution* $\Delta_b = \pi(K + K^{\perp}) \subset TM$ and the *small distribution* $\Delta_s = \pi(K^{\perp}) \subset TM$. These are G-invariant distributions.

For the construction of *reduced spaces*, we will need to consider submanifolds tangent to these distributions. The integrability problem for Δ_s and Δ_b will be discussed in Proposition 3.4, but we make a few observations now. First, note that Δ_s satisfies

$$\Delta_s = \mathrm{Ann}\big(\rho(\mathfrak{h})\big). \tag{13}$$

Since the space of sections of $\rho(\mathfrak{h})$ is generated by closed 1-forms, it follows that Δ_s is an integrable distribution around the points where $\rho(\mathfrak{h})$ has locally constant rank. As we shall see in Section 3.2, in the presence of a moment map $\mu : M \to \mathfrak{h}^*$, Δ_s coincides with the distribution tangent to the level sets, whereas Δ_b is the distribution tangent to the G-orbits of the level sets.

In general, since $\pi(K)$ is the distribution tangent to the G-orbits on M, the G-orbit of any leaf of Δ_s (if smooth) is then a leaf of Δ_b. (Here a *leaf* of a distribution is taken to mean a maximal connected integral submanifold.) In particular, any leaf of Δ_b is G-invariant. These observations allow us to prove the following useful lemma.

Lemma 3.2. *Let $P \subset M$ be a leaf of the big distribution Δ_b on which G acts freely and properly, and suppose $\rho(\mathfrak{h})$ has constant rank along P. Then K and $K \cap K^{\perp}$ both have constant rank along P.*

Proof. Since G acts freely on P, $\pi(K) = \psi(\mathfrak{g})$ has constant rank along P. Further, as $\rho(\mathfrak{h})$ also has constant rank along P, it follows that $\rho(\mathfrak{a}) = K$ has constant rank along P.

From (13) and the discussion following it, we conclude that $\Delta_s|_P \subset TP$ is integrable, defining a regular foliation in P. Moreover, P is the G-orbit of a leaf S of $\Delta_s|_P$.

On the other hand, because ρ is a Courant morphism, we have for all $a \in \mathfrak{a}$,

$$\rho\big([a, a]\big) = \big[\rho(a), \rho(a)\big] = \mathcal{D}\big\langle \rho(a), \rho(a)\big\rangle. \tag{14}$$

Since $[a, a] \in \mathfrak{h}$, it follows that $\rho([a, a])|_{TS} = 0$, so we see that $\langle \rho(a), \rho(a)\rangle$ is constant along S. Hence we obtain an induced inner product on $\mathfrak{a}$, $(a, b) \mapsto \langle \rho(a), \rho(b)\rangle|_S$, whose null space, modulo $\ker \rho|_{\mathfrak{h}}$, maps isomorphically onto $K \cap K^{\perp}$. Hence $K \cap K^{\perp}$ has constant rank along S. But $K \cap K^{\perp}$ is G-invariant, so it must have constant rank over the entire big leaf P. □

Hence, under the assumptions of Lemma 3.2, K and $K \cap K^{\perp}$ are G-invariant vector bundles over P, so we can consider the quotient vector bundle

$$E_{\mathrm{red}} = \frac{K^{\perp}|_P}{K \cap K^{\perp}|_P} \bigg/ G \tag{15}$$

over $M_{\mathrm{red}} := P/G$, the *reduced space*. It is clear that E_{red} inherits a nondegenerate symmetric pairing from the one in E. The next theorem shows that E_{red} carries in fact a Courant algebroid structure. We call it the *reduced Courant algebroid*.

Theorem 3.3. *Let E be an exact Courant algebroid over M and $\rho : \mathfrak{a} \to \Gamma(E)$ be an extended G-action. Let $P \subset M$ be a leaf of Δ_b on which G acts freely and properly, and over which $\rho(\mathfrak{h})$ has constant rank. Then the Courant bracket on E descends to E_{red} and makes it into a Courant algebroid over $M_{\mathrm{red}} = P/G$ with surjective anchor. If K is isotropic then E_{red} is an exact Courant algebroid; in general, it is exact if and only if the following holds along P:*

$$\pi(K) \cap \pi\big(K^{\perp}\big) = \pi\big(K \cap K^{\perp}\big). \tag{16}$$

Proof. By Lemma 3.2, $K^{\perp}$ and $K \cap K^{\perp}$ are G-invariant bundles over P, and hence E_{red} is a vector bundle over $M_{\mathrm{red}} = P/G$ equipped with a nondegenerate symmetric pairing. We will now check that E_{red} inherits a Courant bracket.

Let $\tilde{v}, \tilde{w} \in \Gamma(E)$ be extensions of G-invariant sections of $K^{\perp}$ over P. Note that the bracket $[\tilde{v}, \tilde{w}]$ restricted to P is a section of $K^{\perp}|_P$: for $a \in \mathfrak{a}$, by (C4) we have

$$\begin{aligned}\big\langle \rho(a), [\tilde{v}, \tilde{w}]\big\rangle &= -\big\langle \big[\tilde{v}, \rho(a)\big], \tilde{w}\big\rangle + \pi(\tilde{v})\big\langle \rho(a), \tilde{w}\big\rangle \\ &= \big\langle \big[\rho(a), \tilde{v}\big], \tilde{w}\big\rangle + \pi(\tilde{w})\big\langle \rho(a), \tilde{v}\big\rangle + \pi(\tilde{v})\big\langle \rho(a), \tilde{w}\big\rangle,\end{aligned}$$

which vanishes along P since $\tilde{v}$ is an invariant section of $K^{\perp}$ there, and $\langle \rho(a), \tilde{v}\rangle|_P \equiv \langle \rho(a), \tilde{w}\rangle|_P \equiv 0$. As a result, $[\tilde{v}, \tilde{w}]|_P$ is again a G-invariant section of $K^{\perp}|_P$.

To describe the dependence of $[\tilde{v}, \tilde{w}]|_P$ with respect to the extensions chosen, consider a section of E vanishing along P, i.e., a section of the form sf, where $s \in \Gamma(E)$ and $f \in C^{\infty}(M, \mathbb{R})$, with $f|_P \equiv 0$. Then $[\tilde{v}, sf] = f[\tilde{v}, s] + (\pi(\tilde{v})f)s$, which vanishes upon restriction to P, since $f|_P \equiv 0$ and $\pi(\tilde{v})$ is tangent to P there. Therefore $[\tilde{v}, sf]|_P \equiv 0$. On the other hand, since $[sf, \tilde{w}] = -[\tilde{w}, fs] + \pi^* d(\langle s, \tilde{w}\rangle f)$, it follows that

$$[sf, \tilde{w}]|_P = \langle s, \tilde{w}\rangle \pi^* df|_P. \tag{17}$$

But $df|_P \in \mathrm{Ann}(TP)$ and $\mathrm{Ann}(TP) = \mathrm{Ann}(\pi(K + K^{\perp})|_P) = \{\xi \in T^*M|_P \mid \pi^*\xi \in K \cap K^{\perp}|_P\}$, so the right-hand side of (17) is a section of $K \cap K^{\perp}|_P$. It follows that if $\tilde{v}, \tilde{w}, \tilde{v}', \tilde{w}'$ are sections of E extending G-invariant sections of $K^{\perp}|_P$ such that $(\tilde{v}' - \tilde{v})|_P = 0$ and $(\tilde{w}' - \tilde{w})|_P = 0$, then

$$\big([\tilde{v}, \tilde{w}] - [\tilde{v}', \tilde{w}']\big)\big|_P \in \Gamma\big(K \cap K^{\perp}|_P\big)^G.$$

Hence the bracket on invariant sections of $K^{\perp}|_P$ is well defined modulo sections of $K \cap K^{\perp}|_P$, which means that we have a well defined bracket

$$\Gamma\big(K^{\perp}|_P\big)^G \times \Gamma\big(K^{\perp}|_P\big)^G \to \Gamma(E_{\mathrm{red}}) = \frac{\Gamma(K^{\perp}|_P)^G}{\Gamma(K \cap K^{\perp}|_P)^G}.$$

To see that this bracket descends to a bracket on $\Gamma(E_{\mathrm{red}})$, one must check that if v and w are G-invariant sections in $K^{\perp}|_P$ and $K^{\perp} \cap K|_P$, respectively, then their bracket lies in $K^{\perp} \cap K|_P$. Note that it suffices to check that their bracket lies in $K|_P$, since we already know that it is an invariant section of $K^{\perp}|_P$.

Writing the extensions of v and w to $\Gamma(E)$ as $\tilde{v}$ and $\tilde{w} = \sum f_i \rho(a_i)$, we have

$$[\tilde{v}, \tilde{w}] = \sum_i \big(f_i \big[\tilde{v}, \rho(a_i)\big] + \big(\pi(\tilde{v}) f\big)\rho(a_i)\big)$$
$$= \sum_i \big(-f_i \big[\rho(a_i), \tilde{v}\big] + f_i \pi^* d\big\langle \tilde{v}, \rho(a_i)\big\rangle + \big(\pi(\tilde{v}) f\big)\rho(a_i)\big).$$

Restricting to P, we obtain $[\tilde{v}, \tilde{w}]|_P = \sum_i (\pi(\tilde{v}) f)\rho(a_i)$, which is a section of $K|_P$, as desired. The same conclusion holds for $[\tilde{w}, \tilde{v}]$, and hence we obtain a Courant bracket on $\Gamma(E_{\mathrm{red}})$. This makes E_{red} into a Courant algebroid over $M_{\mathrm{red}} = P/G$ with anchor given by the natural projection, which is clearly surjective.

The Courant algebroid E_{red} is exact if and only if the kernel of its anchor is isotropic. Along P this can be expressed as the condition that $\{v \in K^\perp \colon \pi(v) \in \pi(K)\}$ be isotropic in E. This happens if and only if $\pi(K \cap K^\perp) = \pi(K) \cap \pi(K^\perp)$ in TP. If K itself was isotropic, then $K \subset K^\perp$, and hence the condition would be automatically satisfied. □

We will give explicit examples of reduced Courant algebroids using the construction above in Section 3.3. Note that this construction depends upon a choice of leaf $P \subset M$ of Δ_b. We end this subsection with a discussion of the integrability of the distributions Δ_s and Δ_b.

Suppose that an extended action ρ is such that G acts freely and properly on the *entire manifold* M and $\rho(\mathfrak{h})$ has constant rank *everywhere in* M. Then Δ_s has constant rank by (13), and its integrability follows from the fact that the space of sections of $\rho(\mathfrak{h})$ is generated by closed 1-forms. Hence Δ_s defines a regular foliation of M. The next proposition asserts that, although Δ_b may not have constant rank, it is a generalized integrable distribution in the sense of Sussmann [28], defining a singular foliation of M (i.e., its leaves are smooth immersed submanifolds of varying dimensions):

Proposition 3.4. *Let $\rho : \mathfrak{a} \to \Gamma(E)$ be an extended G-action on an exact Courant algebroid E over M. Assume that the G-action on M is free and proper and that $\rho(\mathfrak{h})$ has constant rank everywhere in M. Let S be a leaf of Δ_s. Then the distribution $TS \cap \psi(\mathfrak{g})$ has constant rank along S, and Δ_b is an integrable generalized distribution.*

Proof. Recall from the proof of Lemma 3.2 that $\langle\rho(a), \rho(b)\rangle$ is constant along S, so $(a, b) \mapsto \langle\rho(a), \rho(b)\rangle|_S$ defines a symmetric bilinear form on $\mathfrak{a}$. We consider the subspace $\mathfrak{h}^\perp \subset \mathfrak{a}$, and its projection to $\mathfrak{g}$, $\pi(\mathfrak{h}^\perp) \subset \mathfrak{g}$. We claim that $\psi(\pi(\mathfrak{h}^\perp)) = TS \cap \psi(\mathfrak{g})$. Indeed, on the one hand,

$$\psi\big(\pi\big(\mathfrak{h}^\perp\big)\big) = \big\{\psi(g) \bigm| \exists a \in \mathfrak{a} \text{ with } g = \pi(a), \text{ and } \rho(a) \in \rho(\mathfrak{h})^\perp\big\}.$$

On the other hand, $TS \cap \psi(\mathfrak{g}) = \{\psi(g) \mid \exists v \in K^\perp, \psi(g) = \pi(v)\}$. But $g = \pi(a)$ for some $a \in \mathfrak{a}$, and $\psi(g) = \pi(\rho(a))$. It follows that $v - \rho(a) \in T^*M$, i.e., $\rho(a) \in K^\perp + T^*M = \rho(\mathfrak{h})^\perp$. Hence $TS \cap \psi(\mathfrak{g})$ has constant rank along S.

Now let $q : M \to M/G$ be the quotient map, which is a surjective submersion. Then $q|_S : S \to M/G$ has constant rank, so $q(S)$ is an (immersed) submanifold of M/G. Then $P = q^{-1}(q(S))$, the G-orbit of S, is an (immersed) submanifold of M whose tangent bundle is $\Delta_b|_P$. Hence Δ_b is integrable is Sussmann's sense. □

As a corollary, observe that the reduced manifolds $q(S) = q(P) = P/G$ are integral submanifolds of the smooth generalized distribution

$$\frac{\pi(K + K^{\perp})}{\pi(K)} \Big/ G \subset T(M/G), \tag{18}$$

showing that this is also integrable in Sussmann's sense. Therefore the orbit space M/G admits a singular foliation by submanifolds which support the reduced Courant algebroids.

Note, however, that we do not need the global integrability of Δ_s and Δ_b for the general construction of reduced Courant algebroids.

3.2. Isotropy action and the reduced Ševera class

In this section we will give an alternative construction of reduced Courant algebroids which clarifies condition (16) and allows us to describe the Ševera class of an exact reduced Courant algebroid. We start by considering the important special case of a trivially extended action, in the sense of Definition 2.12. As the next example shows, in this case condition (16) is precisely the requirement that the action is *isotropic*, i.e. $K \subset K^{\perp}$ or equivalently, in the language of Theorem 2.13, the symmetric form c vanishes and $\Phi = H + \xi_a$ is equivariantly closed.

Example 3.5. Let $\rho : \mathfrak{g} \to \Gamma(E)$ be a trivially extended action of a free and proper action of G on the manifold M, so that $\mathfrak{h} = \{0\}$. Then by Eq. (13), we obtain $\pi(K^{\perp}) = TM$, and in particular, $\Delta_s = \Delta_b = TM$. Hence by Theorem 3.3, we obtain an exact reduced Courant algebroid E_{red} over $M_{\mathrm{red}} = M/G$ if and only if $\pi(K) = \pi(K \cap K^{\perp})$, which occurs if and only if $K \subset K^{\perp}$, since $K \cap T^*M = \{0\}$. This provides an alternate motivation for the requirement in [14] that K be isotropic.

In the case of a trivially extended, isotropic action of a compact Lie group, we obtain the following description of the Ševera class of the reduced Courant algebroid, which appeared implicitly in [8] in the context of gauging the Wess–Zumino term:

Proposition 3.6. *Let G be a compact Lie group acting freely and properly on M, and ρ be a trivially extended, isotropic action on the exact Courant algebroid E over M. Then if $[H] \in H^3(M, \mathbb{R})$ is the Ševera class of E, the reduced Courant algebroid has Ševera class $q_*[\Phi]$, where $\Phi = H + \xi_a$ is the closed equivariant extension induced by ρ, and q_* is the natural isomorphism*

$$H^3_G(M, \mathbb{R}) \xrightarrow{q_*} H^3(B, \mathbb{R}).$$

Furthermore, a splitting $\nabla : TM \to E$ induces a splitting of E_{red} if and only if $\rho(\mathfrak{g}) \subset \nabla(TM)$.

Proof. Since the action is isotropic, the reduced Courant algebroid is exact, so it fits into the exact sequence

$$T^*B \xrightarrow{dq^*} E_{\mathrm{red}} = \big(K^{\perp}/K\big)/G \xrightarrow{dq} TB, \tag{19}$$

where $q : M \to B$ is the quotient map. We can find the reduced Ševera class by choosing an isotropic splitting of this sequence. To find such a splitting, let us first choose a G-invariant splitting of E, so that $E = (TM \oplus T^*M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H)$, with $\rho(a) = X_a + \xi_a$ for $a \in \mathfrak{g}$, and $i_{X_a}H = d\xi_a$, as in Eq. (9). Now let $\theta \in \Omega^1(M, \mathfrak{g})$ be a connection for the principal G-bundle M. The image of the natural map $TB \to TM/G$, $Y \mapsto Y^h$, where Y^h is the horizontal lift of Y, may not lie in $K^\perp/G$, but this holds for the map

$$Y \mapsto Y^h + i_{Y^h}\langle\theta \wedge \xi\rangle, \tag{20}$$

where the 2-form $\langle\theta \wedge \xi\rangle$ is obtained by wedging θ with $\xi \in \Omega^1(M, \mathfrak{g}^*)$ and taking the trace. (This 2-form is invariant since $\mathcal{L}_{X_a}\theta = -\operatorname{ad}_a^*\theta$ and, by Eq. (10), $\mathcal{L}_{X_a}\xi = \operatorname{ad}_a \xi$.) Also note that the image of (20) is isotropic in $K^\perp$ and intersects K trivially. Therefore (20) induces a map

$$\nabla : TB \to E_{\mathrm{red}}$$

which is an isotropic splitting of (19). Now the induced 3-form on $TB \oplus T^*B$ is given by

$$\begin{aligned}
\tilde{H}(X, Y, Z) &= 2\big\langle\big[\nabla(X), \nabla(Y)\big], \nabla(Z)\big\rangle \\
&= 2\big\langle\big[X^h + i_{X^h}\langle\theta \wedge \xi\rangle, Y^h + i_{Y^h}\langle\theta \wedge \xi\rangle\big]_H, Z^h + i_{Z^h}\langle\theta \wedge \xi\rangle\big\rangle \\
&= 2\big\langle\big[X^h, Y^h\big]_{H+d\langle\theta\wedge\xi\rangle}, Z^h\big\rangle \\
&= \big(h + \langle F \wedge \xi\rangle\big)(X, Y, Z),
\end{aligned}$$

where in the last equality, h is the basic component of H, $F \in \Omega^2(M, \mathfrak{g})$ is the curvature of θ, and we have used the fact that, when evaluated on horizontal vectors,

$$d\langle\theta \wedge \xi\rangle\big(X^h, Y^h, Z^h\big) = \langle F \wedge \xi\rangle\big(X^h, Y^h, Z^h\big).$$

The mapping obtained here, which sends $H + \xi_a$ to the closed form $h + \langle F \wedge \xi\rangle \in \Omega^3(B, \mathbb{R})$ on the base, is exactly the form-level push-down isomorphism in equivariant cohomology:

$$H^3_G(M, \mathbb{R}) \xrightarrow{q_*} H^3(B, \mathbb{R}).$$

So the curvature of the reduced exact Courant algebroid is precisely the push-forward of the equivariant extension of the original curvature induced by the extended action.

Note also that the splitting of E_{red} used to calculate $\tilde{H}$ depends on the choice of connection unless $\xi = 0$, in which case it is naturally induced from the original splitting of E. □

We now explain how one can use these results about trivially extended actions to tackle the general case. The key observation is that there is an alternate construction of reduced Courant algebroids which consists of two steps: first a restriction to a small leaf $S \subset M$, and then a reduction through a trivially extended action of a smaller group that acts on S.

The first step is based on the fact exact Courant algebroids may always be *pulled back* to submanifolds:

Lemma 3.7. *Let $\iota : S \hookrightarrow M$ be a submanifold of a manifold equipped with an exact Courant algebroid E. Then the vector bundle*

$$E_S := \frac{(\operatorname{Ann}(TS))^\perp}{\operatorname{Ann}(TS)} = \frac{\pi^{-1}(TS)}{\operatorname{Ann}(TS)} \tag{21}$$

inherits the structure of an exact Courant algebroid over S with Ševera class $\iota^*[H]$, where $[H]$ is the class of E.

Proof. The subbundle $\mathrm{Ann}(TS) \subset T^*M \subset E$ is isotropic, so E_S has a natural nondegenerate symmetric bilinear form. It inherits a Courant bracket by restriction, as in the proof of Theorem 3.3, and a simple dimension count shows that this Courant algebroid must be exact.

If a splitting $TM \to E$ were chosen, rendering E isomorphic to $(TM \oplus T^*M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H)$ with $H \in \Omega^3_{\mathrm{cl}}(M)$, then $\pi^{-1}(TS) = TS + T^*M$, and we would obtain a natural splitting

$$E_{\mathrm{red}} = TS \oplus T^*M/\mathrm{Ann}(TS) = TS \oplus T^*S.$$

With this identification, the 3-form twisting the Courant algebroid structure on $TS \oplus T^*S$ is simply the pull-back ι^*H. □

Let us consider an extended action on an exact Courant algebroid E over M, and let S be a leaf of the small distribution Δ_s. As we saw from (14), $\langle\rho(a), \rho(b)\rangle$ is constant along a small leaf S and induces a symmetric bilinear form on the Courant algebra $\mathfrak{a}$, for which $\mathfrak{h}$ is isotropic. Therefore we may define $\mathfrak{a}_s = \mathfrak{h}^\perp$ and $\mathfrak{g}_s = \pi(\mathfrak{a}_s)$, noting that $\mathfrak{a}_s$ is closed under the Courant bracket. This implies that $\mathfrak{g}_s$ is a Lie subalgebra of $\mathfrak{g}$, which we call the *isotropy subalgebra*, and it inherits a symmetric bilinear form $c_s \in S^2(\mathfrak{g}_s^*)$ by construction. Therefore we obtain the sub-Courant algebra

$$0 \to \mathfrak{h} \to \mathfrak{a}_s \xrightarrow{\pi} \mathfrak{g}_s \to 0,$$

which is mapped via the extended action ρ into $\pi^{-1}(TS)$. Quotienting by $\mathfrak{h}$, we obtain a trivially extended action ρ_s of the isotropy subalgebra on the pull-back Courant algebra E_S over S,

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \mathfrak{g}_s & \xrightarrow{\pi} & \mathfrak{g}_s & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \rho_s} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & \Gamma(T^*S) & \longrightarrow & \Gamma(E_S) & \longrightarrow & \Gamma(TS) & \longrightarrow & 0 \end{array}$$

which satisfies $\langle\rho_s(a), \rho_s(b)\rangle = c_s(a,b)$ by construction. Note that the underlying group action on S is by the subgroup $G_s \subset G$ stabilizing S, which we call the *isotropy subgroup*. Also there is a natural isomorphism $S/G_s \to P/G$ if P is a leaf of Δ_b containing S and satisfying the conditions of Theorem 3.3 (see Proposition 3.4).

These arguments show that after pull-back to S, we obtain a trivially extended action as in Example 3.5. The quotient of this pull-back turns out to be naturally isomorphic to the quotient Courant algebroid E_{red} constructed in Theorem 3.3, and we conclude that E_{red} is exact if and only if the action ρ_s is isotropic, i.e. $\rho_s(\mathfrak{g}_s) \subset \rho_s(\mathfrak{g}_s)^\perp$.

Proposition 3.8. *Let P be as in Theorem* 3.3*, and let $\iota : S \hookrightarrow P$ be a leaf of Δ_s. Then the reduced Courant algebroid E_{red} over P/G is naturally isomorphic to the quotient of the pull-back E_S by the isotropy action ρ_s. In particular, E_{red} is exact if and only if ρ_s is isotropic, i.e. $c_s \in S^2(\mathfrak{g}_s^*)$ vanishes.*

Proof. The image of the isotropy action ρ_s in E_S is given by

$$K_s = \frac{K \cap (K^\perp + T^*M)}{K \cap T^*M}\bigg|_S \subset E_S = \frac{K^\perp + T^*M}{K \cap T^*M}\bigg|_S.$$

Then the reduced Courant algebroid over S/G_s is the G_s quotient of the bundle

$$\frac{K_s^\perp}{K_s \cap K_s^\perp} = \frac{(K^\perp + K \cap T^*M)/K \cap T^*M}{(K \cap K^\perp + K \cap T^*M)/K \cap T^*M}\bigg|_S,$$

which is canonically isomorphic to $E_{\mathrm{red}} = (\frac{K^\perp}{K \cap K^\perp}|_P)/G$ as a Courant algebroid. Since ρ_s is a trivially extended action, we conclude from Example 3.5 that E_{red} is exact if and only if K_s is isotropic in E_S, a condition equivalent to the requirement that $\tilde{K} \subset E$ is isotropic along P, where

$$\widetilde{K} = K \cap \big(K^\perp + T^*M\big). \qquad \square \tag{22}$$

The previous proposition, combined with Proposition 3.6 and Lemma 3.7, provides a description of the Ševera class of any exact reduced Courant algebroid: simply pull back the 3-form to the leaf S of Δ_s and apply Proposition 3.6 for the isotropy action ρ_s.

In the presence of a moment map $\mu : M \to \mathfrak{h}^*$ for the generalized action, the moment map condition $d(\mu_h) = \rho(h)$ implies that

$$\ker(d\mu) = \mathrm{Ann}\big(\rho(\mathfrak{h})\big) = \Delta_s,$$

so that the leaves of the small distribution Δ_s are precisely the level sets $\mu^{-1}(\lambda)$ of the moment map. Similarly the leaves of the big distribution are inverse images $\mu^{-1}(\mathcal{O}_\lambda)$ of orbits $\mathcal{O}_\lambda \subset \mathfrak{h}^*$ of the action of G. The small leaf $S = \mu^{-1}(\lambda)$ then has isotropy Lie algebra $\mathfrak{g}_s = \mathfrak{g}_\lambda$, which is the Lie algebra of G_λ, the subgroup stabilizing λ under the action of G on $\mathfrak{h}^*$. Applying Theorem 3.3 together with Proposition 3.8, we obtain the following formulation of the reduction procedure:

Proposition 3.9 *(Moment map reduction). Let the extended action ρ on the Courant algebroid E have moment map μ. Then the reduced Courant algebroid associated to the regular value $\lambda \in \mathfrak{h}^*$ is obtained via pull-back E_S along $\iota : S = \mu^{-1}(\lambda) \hookrightarrow M$, followed by reduction by the isotropy action ρ_λ of G_λ on the level set, which we assume is free and proper. The result is an exact Courant algebroid if and only if ρ_λ is isotropic, i.e. the induced symmetric form $c_\lambda \in S^2(\mathfrak{g}_\lambda^*)$ vanishes.*

3.3. Examples

In this section we will provide some examples of Courant algebroid reduction, illustrating the results of Sections 3.1 and 3.2.

Example 3.10. Even a trivial group action may be extended by 1-forms; consider the extended action $\rho : \mathbb{R} \to \Gamma(E)$ on an exact Courant algebroid E over M given by $\rho(1) = \xi$ for some closed 1-form ξ. Then $K = \langle \xi \rangle$ and $K^\perp = \{v \in E\colon\ \pi(v) \in \mathrm{Ann}(\xi)\}$ which induces the distribution $\Delta_b = \Delta_s = \mathrm{Ann}(\xi) \subset TM$, which is integrable wherever ξ is nonzero. Since the group action is trivial, a reduced Courant algebroid is simply a choice of integral submanifold $\iota : S \hookrightarrow M$ for ξ

together with the pull-back exact Courant algebroid $E_{\mathrm{red}} = E_S = K^{\perp}/K$, as in Lemma 3.7. The Ševera class in this case is the pull-back to S of the class of E.

Example 3.11. At another extreme, consider a free and proper action of G on M, with infinitesimal action $\psi : \mathfrak{g} \to \Gamma(TM)$, and extend it trivially by inclusion to a split Courant algebroid $(TM \oplus T^*M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H)$ such that the splitting is preserved by the action. By Eq. (9), this is equivalent to the requirement that H is an invariant basic form.

Then $K = \psi(\mathfrak{g})$ and $K^{\perp} = TM \oplus \mathrm{Ann}(K)$, so that $\Delta_s = \Delta_b = TM$ and the reduced Courant algebroid is

$$TM/K \oplus \mathrm{Ann}(K) = TB \oplus T^*B,$$

where $B = M/G$ is the quotient and the 3-form twisting the Courant bracket on B is the push-down of the basic form H.

The next example shows explicitly that a trivial twisting $[H] = 0$ may give rise to a cohomologically nontrivial reduced Courant algebroid.

Example 3.12. Consider $M = S^3 \times S^1$ as an S^1-bundle over $S^2 \times S^1$, where the S^1-action on the first factor of M generates the Hopf bundle $S^3 \to S^2$, and the action on the second factor is trivial. We denote the infinitesimal generator of the action on M by ∂_t. If ξ is a volume form in S^1, then $\rho(1) = \partial_t + \xi$ defines a trivially extended, isotropic S^1-action on the Courant algebroid $TM \oplus T^*M$, with $H = 0$. By Proposition 3.6, the reduced Courant algebroid over $S^2 \times S^1$ has curvature $F \wedge \xi$, where F is the Chern class of the Hopf fibration, so the reduced Ševera class is nontrivial.

As we saw in Proposition 3.6, a reduced Courant algebroid may not inherit a canonical splitting. The next example illustrates a situation where the reduced Courant algebroids are naturally split.

Example 3.13. One situation where E_{red} always inherits a splitting is when E is equipped with a G-invariant splitting ∇ and the action ρ is split, in the sense that there is a splitting s for $\pi : \mathfrak{a} \to \mathfrak{g}$ making the diagram commutative:

$$\begin{array}{ccc} \mathfrak{a} & \xleftarrow{\;s\;} & \mathfrak{g} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \psi} \\ \Gamma(E) & \xleftarrow{\;\nabla\;} & \Gamma(TM). \end{array} \tag{23}$$

In this case, the image distribution $\rho(\mathfrak{a}) = K$ decomposes as $K = K_T \oplus K_{T^*}$, with $K_T \subset TM$ and $K_{T^*} \subset T^*M$, and hence we have the pointwise identification

$$\frac{K^{\perp}}{K \cap K^{\perp}} = \left(\frac{\mathrm{Ann}(K_T)}{\mathrm{Ann}(K_T) \cap K_{T^*}}\right) \oplus \left(\frac{\mathrm{Ann}(K_{T^*})}{\mathrm{Ann}(K_{T^*}) \cap K_T}\right). \tag{24}$$

Upon restriction to a leaf P and quotient by G, the distribution (24) agrees with $TM^*_{\mathrm{red}} \oplus TM_{\mathrm{red}}$, since $\mathrm{Ann}(K_{T^*})/(\mathrm{Ann}(K_{T^*}) \cap K_T) = \Delta_s/\rho(\mathfrak{g}_s)$. Hence E_{red} is split. The curvature H of the given splitting for E is then basic, and the curvature for E_{red} is simply the pull-back to S followed by push-down to S/G_s.

If we are in the situation above, where the action is split, one has a natural trivially extended G-action on M coming from $\rho \circ s$. Assuming that G acts freely and properly on all of M, we may form the quotient Courant algebroid E^1_{red} over M/G, which is exact since $\rho \circ s$ is isotropic. Assuming that $\rho(\mathfrak{h})$ had constant rank on M, then as we saw in Section 3.1, M/G inherits a generalized foliation; the pull-back of E^1_{red} to a leaf of this foliation would then recover the reduced Courant algebroid E_{red} over M_{red} constructed as before.

An example of such a split action, where the reduced Courant algebroid may be obtained in two equivalent ways, is the case of a symplectic action, as introduced in Example 2.16.

Example 3.14. Let (M, ω) be a symplectic manifold and consider the extended G-action $\rho : \mathfrak{g} \oplus \mathfrak{g} \to \Gamma(TM \oplus T^*M)$ with curvature $H = 0$ defined in Example 2.16. This is clearly a split action in the above sense.

Let $\psi : \mathfrak{g} \to \Gamma(TM)$ be the infinitesimal action and $\psi(\mathfrak{g})^\omega$ denote the symplectic orthogonal of the image distribution $\psi(\mathfrak{g})$. Then the extended action has image

$$K = \psi(\mathfrak{g}) \oplus \omega\big(\psi(\mathfrak{g})\big),$$

so that the orthogonal complement is

$$K^\perp = \psi(\mathfrak{g})^\omega \oplus \operatorname{Ann}\big(\psi(\mathfrak{g})\big).$$

Then the big and small distributions on M are

$$\Delta_s = \psi(\mathfrak{g})^\omega,$$
$$\Delta_b = \psi(\mathfrak{g})^\omega + \psi(\mathfrak{g}).$$

If the action is Hamiltonian, with moment map $\mu : M \to \mathfrak{g}^*$, then Δ_s is the tangent distribution to the level sets $\mu^{-1}(\lambda)$ while Δ_b is the tangent distribution to the sets $\mu^{-1}(\mathcal{O}_\lambda)$, for $\mathcal{O}_\lambda$ a coadjoint orbit containing λ. Therefore we see that the reduced Courant algebroid is simply $TM_{\text{red}} \oplus T^*M_{\text{red}}$ with $H = 0$, for the usual symplectic reduced space $M_{\text{red}} = \mu^{-1}(\mathcal{O}_\lambda)/G = \mu^{-1}(\lambda)/G_\lambda$.

Since the action is split, we may also observe, assuming that G acts freely and properly on M, that the quotient M/G is foliated via (18) by the possible reduced spaces. This generalized distribution is given in this case by

$$\frac{\psi(\mathfrak{g})^\omega + \psi(\mathfrak{g})}{\psi(\mathfrak{g})} \Big/ G = dq\big(\psi(\mathfrak{g})^\omega\big) \subset T(M/G),$$

where $q : M \to M/G$ is the quotient map. This is precisely the distribution defined by the image of the Poisson tensor $\Pi : T^*(M/G) \to T(M/G)$ induced by ω (recall that $\Pi(df) = dq(X_{q^*f})$, where X_{q^*f} is the Hamiltonian vector field for q^*f). So a reduced manifold for the extended action is just a symplectic leaf of M/G.

Finally, we present an example of a reduced Courant algebroid which is not exact.

Example 3.15. Let $\rho : \mathfrak{s}^1 \to E$ be a trivially extended S^1 action which is not isotropic, i.e. $\langle \rho(1), \rho(1) \rangle \neq 0$. Hence the reduced manifold for this action is just M/S^1 and the reduced algebroid is $E_{\mathrm{red}} = (K^\perp/(K \cap K^\perp))/S^1$. However, $K \cap K^\perp = \{0\}$ and so E_{red} is odd-dimensional; hence it is not an exact Courant algebroid.

4. Reduction of Dirac structures

A *Dirac structure* [5,21] on a manifold M equipped with exact Courant algebroid E is a maximal isotropic subbundle $D \subset E$ whose sections are closed under the Courant bracket. This last requirement is referred to as the *integrability* condition for D. When the Courant algebroid is split, with curvature $H \in \Omega^3_{\mathrm{cl}}(M)$, these are usually referred to as H*-twisted* Dirac structures [26].

For $H = 0$, examples of Dirac structures on M include closed 2-forms and Poisson bivector fields (in these cases D is simply the graph of the defining tensor, viewed either as a map $\omega : TM \to T^*M$ or $\Pi : T^*M \to TM$) as well as involutive regular distributions $F \subset TM$, in which case $D = F \oplus \mathrm{Ann}(F)$.

In the presence of an extended action of a connected Lie group G on the Courant algebroid E, one may consider Dirac structures which are G-invariant subbundles of E, a condition equivalent to the following.

Definition 4.1. A Dirac structure $D \subset E$ is preserved by an extended action ρ if and only if $[\rho(\mathfrak{a}), \Gamma(D)] \subset \Gamma(D)$.

In this section we explain how a Dirac structure which is preserved by an extended action may be transported from a Courant algebroid E to its reduction E_{red}.

4.1. Reduction procedure

To see how Dirac structures are transported under Courant reduction, we first explain the map at the level of linear algebra. Suppose that E is a real vector space equipped with a nondegenerate symmetric bilinear form of split signature, and suppose that an isotropic subspace $K \subset E$ is given. Then the Courant reduction along K is defined to be $E_{\mathrm{red}} = K^\perp/K$. Furthermore, there is a *canonical relation* between E and E_{red}, i.e. a maximal isotropic subspace

$$\varphi_K = \big\{ \big(x, [x]\big) \in \overline{E} \times E_{\mathrm{red}} \colon\ x \in K^\perp \big\},$$

where $\overline{E}$ denotes E with negative symmetric form. If $D \subset E$ is a Dirac structure (i.e. a maximal isotropic subspace), view it as a relation $D \subset \{0\} \times E$. Then composition (as a relation) with φ_K defines a Dirac structure in E_{red}, given by

$$D_{\mathrm{red}} := \varphi_K \circ D = \frac{D \cap K^\perp + K}{K} \subset E_{\mathrm{red}}.$$

In this way we obtain a reduction map on Dirac structures. This is entirely analogous to the reduction of Lagrangian subspaces under a symplectic reduction, as described by Weinstein [30] using canonical relations in the symplectic category.

Now if E is an exact Courant algebroid and $K = \rho(\mathfrak{a})$ is the image of an extended action which is isotropic along a big leaf $P \subset M$, then any G-invariant Dirac structure D along P gives rise to the following *reduced Dirac structure*, assuming the result is a smooth bundle:

$$D_{\mathrm{red}} = \frac{(D \cap K^{\perp} + K)|_P}{K|_P} \Big/ G \subset E_{\mathrm{red}}.$$

Note that D_{red} is smooth if $D \cap K^{\perp}$ (or equivalently $D \cap K$) has constant rank over P. For the proof that D_{red} is integrable, see Theorem 4.2.

If the extended action is not isotropic, the procedure just described must be modified. In this case we use the result of Proposition 3.8 that the exact reduced Courant algebroid E_{red} can be constructed by first pulling E back to a leaf $S \subset M$ of Δ_s and then taking the quotient by the isotropic action ρ_s. Over the leaf S, the isotropic subbundle $\rho(\mathfrak{h}) = K \cap T^*M$ determines a map of Dirac structures from $E|_S$ to the pull-back Courant algebroid E_S. This is a generalization of the pull-back of Dirac structures defined in [4]. After pull-back, the isotropy action $\rho_s(\mathfrak{g}_s) \subset E_S$ determines a map of Dirac structures from E_S to E_{red}. This is a generalization of the Dirac push-forward [4]. The composition of these maps takes any G-invariant Dirac structure D along P to

$$D_{\mathrm{red}} = \frac{(D \cap \widetilde{K}^{\perp} + \widetilde{K})|_P}{\widetilde{K}|_P} \Big/ G \subset E_{\mathrm{red}}. \tag{25}$$

where $\widetilde{K} = K \cap (K^{\perp} + T^*M) \subset E$, as defined in (22). As a result, the reduced Dirac structure is obtained by the same procedure as in the isotropic case, applied to $\widetilde{K}$ instead of K. We now show that D_{red}, when smooth as a vector bundle, is automatically integrable.

Theorem 4.2. *Let E, ρ, and P be as in Theorem* 3.3, *and such that E_{red} is exact over $M_{\mathrm{red}} = P/G$. Let the action ρ preserve a Dirac structure $D \subset E$. Then if D_{red}, as described above* (25), *is a smooth subbundle* (*e.g. if $D \cap \widetilde{K}$ has constant rank*), *it defines a Dirac structure on the reduction M_{red}.*

Proof. The only property of D_{red} that remains to be checked is integrability. To do so, we first observe that the Courant bracket on $E_{\mathrm{red}} = (\widetilde{K}^{\perp}/\widetilde{K})|_P/G$ admits the following description, equivalent to the one given in Theorem 3.3. Given sections v_1, v_2 of E_{red}, let us consider representatives in $\Gamma(\widetilde{K}^{\perp}|_P)^G$, still denoted by v_1, v_2. Then extend them to sections $\tilde{v}_1, \tilde{v}_2$ of E over M, and define $[v_1, v_2]$ as $[\tilde{v}_1, \tilde{v}_2]|_P$. Similarly to Theorem 3.3, one can show that $[\tilde{v}_1, \tilde{v}_2]|_P \in \Gamma(\widetilde{K}^{\perp}|_P)^G$, and that different choices of extensions change the bracket by invariant sections of $\widetilde{K}$ over P. Also, the bracket between elements in $\Gamma(\widetilde{K}|_P)^G$ and $\Gamma(\widetilde{K}^{\perp}|_P)^G$ remains in $\Gamma(\widetilde{K}|_P)^G$, so there is an induced bracket on E_{red}. This bracket agrees with the one defined in Theorem 3.3.

Let $v_1, v_2 \in \Gamma((D \cap \widetilde{K}^{\perp} + \widetilde{K})|_P)^G$, thought of as representing sections of D_{red}. We note that, around points of P where $D \cap \widetilde{K}^{\perp}|_P$ has locally constant rank, we can write $v_i = v_i' + v_i''$, where v_i' is an invariant local section of $D \cap \widetilde{K}^{\perp}|_P$, and v_i'' is an invariant local section of $\widetilde{K}|_P$. Then the bracket of v_1, v_2 is

$$[v_1' + v_1'', v_2' + v_2''] = [v_1', v_2'] + [v_1'', v_2'] + [v_1', v_2''] + [v_1'', v_2''].$$

Note that the last three terms on the right-hand side are in $\Gamma(\tilde{K}|_P)^G$. As for the first term, we know that it lies in $\widetilde{K}^{\perp}|_P$. But since D is a vector bundle over M, we can locally extend v_i' to sections of D away of P and, using these extensions to compute the bracket, we see that $[v_1', v_2'] \in \Gamma(D|_P)$, since D is closed under the bracket. As a result, we conclude that $[v_1, v_2]$ is in $(D \cap \widetilde{K}^{\perp} + \widetilde{K})|_P$ around points where $D \cap \widetilde{K}^{\perp}|_P$ is locally a bundle.

Since the points of P where $D \cap \widetilde{K}^{\perp}|_P$ has locally constant rank is an open dense set, the argument above shows that for $v_1, v_2 \in \Gamma(D_{\mathrm{red}})$, $[v_1, v_2]$ lies in D_{red} over all points in an open dense subset of P/G. But since D_{red} is smooth, this implies that $[v_1, v_2] \in \Gamma(D_{\mathrm{red}})$, hence D_{red} is integrable. □

The reduction of Dirac structures works in the same way for *complex* Dirac structures, provided one replaces K by its complexification $K_{\mathbb{C}} = K \otimes \mathbb{C}$.

5. Reduction of generalized complex structures

A *generalized complex structure* [9,11] on a manifold M equipped with exact Courant algebroid E is a complex structure on the vector bundle E which is orthogonal with respect to the bilinear pairing and whose $+i$-eigenbundle is closed under the bracket. If the Courant algebroid is split, with curvature $H \in \Omega^3_{\mathrm{cl}}(M)$, a generalized complex structure on E is called an *H-twisted* generalized complex structure on M.

Since a generalized complex structure is orthogonal, its $+i$-eigenbundle $L \subset E \otimes \mathbb{C} = E_{\mathbb{C}}$ is a maximal isotropic subbundle. Therefore a generalized complex structure on E is equivalent to a complex Dirac structure L satisfying

$$L \cap \overline{L} = \{0\}. \tag{26}$$

The *type* of a generalized complex structure at a point $p \in M$ is the complex dimension of the kernel of the projection $\pi : L \to T_{\mathbb{C}}M$ at p. Two basic examples of generalized complex structures on a manifold M (with $H = 0$) arise as follows:

- Let $I : TM \to TM$ be a complex structure on M. Then it induces a generalized complex structure on M by

$$\mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}.$$

  The associated Dirac structure is $L = TM^{0,1} \oplus T^*M^{1,0}$, which has type n.
- Let $\omega : TM \to T^*M$ be a symplectic structure. The induced generalized complex structure is

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

  The associated Dirac structure is $L = \{X - i\omega(X):\ X \in T_{\mathbb{C}}M\}$, and the type is zero.

A generalized complex structure on M^{2n} is of *complex type* if it has type n at all points, and it is of *symplectic type* if it has type zero at all points. The reader is referred to [9] for more details concerning generalized complex structures.

5.1. Reduction procedure

Throughout this section, $\rho : \mathfrak{a} \to \Gamma(E)$ denotes an extended action of a connected Lie group G on an exact Courant algebroid E over a manifold M. Let $K = \rho(\mathfrak{a})$, and let $K_{\mathbb{C}} = K \otimes \mathbb{C}$. We fix a leaf $P \hookrightarrow M$ of the distribution Δ_b as in Theorem 3.3 and assume that the reduced Courant algebroid E_{red} over P/G is exact, which amounts to the assumption that $\widetilde{K} = K \cap (K^{\perp} + T^*M)$ is isotropic along P.

Suppose that the extended action ρ preserves a generalized complex structure $\mathcal{J}$ on E, i.e., that the associated Dirac structure $L \subset E_{\mathbb{C}}$ is invariant. We consider its reduction to E_{red}:

$$L_{\mathrm{red}} = \frac{(L \cap \widetilde{K}_{\mathbb{C}}^{\perp} + \widetilde{K}_{\mathbb{C}})|_P}{\widetilde{K}_{\mathbb{C}}|_P} \Big/ G. \tag{27}$$

If L_{red} is a smooth vector bundle, then it determines a generalized complex structure on E_{red} if and only if it satisfies $L_{\mathrm{red}} \cap \overline{L}_{\mathrm{red}} = \{0\}$.

Lemma 5.1. *The distribution L_{red} satisfies $L_{\mathrm{red}} \cap \overline{L}_{\mathrm{red}} = \{0\}$ if and only if*

$$\mathcal{J}\widetilde{K} \cap \widetilde{K}^{\perp} \subset \widetilde{K} \quad \textit{over } P. \tag{28}$$

Proof. It is clear from (27) that $L_{\mathrm{red}} \cap \overline{L}_{\mathrm{red}} = \{0\}$ over the reduced manifold if and only if

$$\big(L \cap \widetilde{K}_{\mathbb{C}}^{\perp} + \widetilde{K}_{\mathbb{C}}\big) \cap \big(\overline{L} \cap \widetilde{K}_{\mathbb{C}}^{\perp} + \widetilde{K}_{\mathbb{C}}\big) \subset \widetilde{K}_{\mathbb{C}} \quad \text{over } P. \tag{29}$$

Hence, we must prove that conditions (28) and (29) are equivalent.

We first prove that (28) implies (29). Let $v \in (L \cap \widetilde{K}_{\mathbb{C}}^{\perp} + \widetilde{K}_{\mathbb{C}}) \cap (\overline{L} \cap \widetilde{K}_{\mathbb{C}}^{\perp} + \widetilde{K}_{\mathbb{C}})$ over a given point. Without loss of generality we can assume that v is real. Since $v \in L \cap \widetilde{K}_{\mathbb{C}}^{\perp} + \widetilde{K}_{\mathbb{C}}$, we can find $v_L \in L \cap \widetilde{K}_{\mathbb{C}}^{\perp}$ and $v_{\widetilde{K}} \in \widetilde{K}_{\mathbb{C}}$ such that $v = v_L + v_{\widetilde{K}}$. Taking conjugates, we get that $v = \overline{v_L} + \overline{v_{\widetilde{K}}}$, hence $v_L - \overline{v_L} = \overline{v_{\widetilde{K}}} - v_{\widetilde{K}}$. Applying $-i\mathcal{J}$, we obtain

$$v_L + \overline{v_L} = -i\mathcal{J}(\overline{v_{\widetilde{K}}} - v_{\widetilde{K}}).$$

The left-hand side lies in $\widetilde{K}^{\perp}$ while the right-hand side lies in $\mathcal{J}\widetilde{K}$. It follows from (28) that $v_L + \overline{v_L} \in \widetilde{K}$, hence $v = \frac{1}{2}(v_L + \overline{v_L} + v_{\widetilde{K}} + \overline{v_{\widetilde{K}}}) \in \widetilde{K}$, as desired.

Conversely, if (28) does not hold, i.e., there is $v \in \mathcal{J}\widetilde{K} \cap \widetilde{K}^{\perp}$ with $v \notin \widetilde{K}$, then $v - i\mathcal{J}v \in L \cap \widetilde{K}_{\mathbb{C}}^{\perp}$ and $v + i\mathcal{J}v \in \overline{L} \cap \widetilde{K}_{\mathbb{C}}^{\perp}$. Since $v \in \mathcal{J}\widetilde{K}$ and $\mathcal{J}v \in \widetilde{K}$, it follows that $v \in L \cap \widetilde{K}_{\mathbb{C}}^{\perp} + \widetilde{K}_{\mathbb{C}}$ and $v \in \overline{L} \cap \widetilde{K}_{\mathbb{C}}^{\perp} + \widetilde{K}_{\mathbb{C}}$, showing that $(L \cap \widetilde{K}_{\mathbb{C}}^{\perp} + \widetilde{K}_{\mathbb{C}}) \cap (\overline{L} \cap \widetilde{K}_{\mathbb{C}}^{\perp} + \widetilde{K}_{\mathbb{C}}) \not\subset \widetilde{K}_{\mathbb{C}}$. This concludes the proof. □

If the Dirac reduction of the $+i$-eigenbundle of a generalized complex structure $\mathcal{J}$ on E defines a generalized complex structure on E_{red}, then we denote it by $\mathcal{J}^{\mathrm{red}}$. We now present a situation where this occurs.

Theorem 5.2. *Let E, ρ, and P be as in Theorem* 3.3, *and such that E_{red} is exact over $M_{\mathrm{red}} = P/G$. If the action preserves a generalized complex structure $\mathcal{J}$ on E and $\mathcal{J}K = K$ over P then $\mathcal{J}$ reduces to E_{red}.*

Proof. Let us consider the isotropic distribution in the exact Courant algebroid given by

$$L' = \frac{(L \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}} \cap K_{\mathbb{C}}^{\perp})|_P}{(K_{\mathbb{C}} \cap K_{\mathbb{C}}^{\perp})|_P} \Big/ G \subset E_{\mathrm{red}} \otimes \mathbb{C}. \tag{30}$$

One can check that $L' \subset L_{\mathrm{red}}$, so in order to show that L' and L_{red} coincide, it suffices to show that L' is maximal isotropic. This is what we will check now.

Since $\mathcal{J} K^{\perp} = K^{\perp}$ over P, it follows that $K_{\mathbb{C}}^{\perp} = L \cap K_{\mathbb{C}}^{\perp} + \overline{L} \cap K_{\mathbb{C}}^{\perp}$. Hence we have

$$\frac{L \cap K_{\mathbb{C}}^{\perp} + \overline{L} \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}} \cap K_{\mathbb{C}}^{\perp}}{K_{\mathbb{C}} \cap K_{\mathbb{C}}^{\perp}} = \frac{K_{\mathbb{C}}^{\perp}}{K_{\mathbb{C}} \cap K_{\mathbb{C}}^{\perp}} \quad \text{along } P.$$

After quotienting by G, this implies that $L' + \overline{L'} = E_{\mathrm{red}} \otimes \mathbb{C}$, showing that L' is maximal. Hence $L_{\mathrm{red}} = L'$. Also note that $L \cap K_{\mathbb{C}}^{\perp}$ has constant rank over P and, since $K_{\mathbb{C}} \cap K_{\mathbb{C}}^{\perp}$ is a bundle over P, this implies that L' as defined in (30) is smooth.

Finally, in order to conclude that L_{red} induces a generalized complex structure we must check that condition (28) in Lemma 5.1 holds:

$$\mathcal{J}\widetilde{K} \cap \widetilde{K}^{\perp} = K \cap \big(K^{\perp} + \mathcal{J} T^*M\big) \cap \big(K^{\perp} + K \cap T^*M\big) \subset K \cap \big(K^{\perp} + K \cap T^*M\big) = \widetilde{K}. \qquad \square$$

Corollary 5.3. *If the hypotheses of the previous theorem hold and the extended action has a moment map $\mu : M \to \mathfrak{h}^*$, then the reduced Courant algebroid over $\mu^{-1}(O_\lambda)/G$ obtains a generalized complex structure.*

Theorem 5.2 uses the compatibility condition $\mathcal{J} K = K$ for the reduction of $\mathcal{J}$. We now observe that the reduction procedure also works in an extreme opposite situation.

Proposition 5.4. *Let E, ρ, and P be as in Theorem* 3.3*, and such that $\rho(\mathfrak{a}) = K$ is isotropic over P. If ρ preserves J, and $\langle\cdot,\cdot\rangle$ is a nondegenerate pairing between K and $\mathcal{J} K$, then $\mathcal{J}$ reduces to E_{red}.*

Proof. As K is isotropic over P, the reduced Courant algebroid is exact and $\widetilde{K} = K$. The nondegeneracy assumption implies that $\mathcal{J} K \cap K^{\perp} = \{0\}$, and it follows that $L \cap K_{\mathbb{C}}^{\perp}$ is a bundle and the Dirac reduction of L is smooth. Finally, (28) holds trivially. $\square$

5.2. Symplectic structures

We now present two examples of reduction obtained from a symplectic manifold (M, ω): First, we show that ordinary symplectic reduction is a particular case of our construction; the second example illustrates how one can obtain a type 1 generalized complex structure as the reduction of an ordinary symplectic structure. In both examples, the initial Courant algebroid is just $TM \oplus T^*M$ with $H = 0$.

Example 5.5 *(Ordinary symplectic reduction).* Let (M, ω) be a symplectic manifold, and let $\mathcal{J}_\omega$ be the generalized complex structure associated with ω. Following Example 2.16 and keeping the same notation, consider a symplectic G-action on M, regarded as an extended action. It is clear that $\mathcal{J}_\omega K = K$, so we are in the situation of Theorem 5.2.

Following Example 3.14, let S be a leaf of the distribution $\Delta_s = \psi(\mathfrak{g})^\omega$. Since K splits as $K_T \oplus K_{T^*}$, the reduction procedure of Theorem 4.2 in this case amounts to the usual pull-back of ω to S, followed by a Dirac push-forward to $S/G_s = M_{\mathrm{red}}$. If the symplectic action admits a moment map $\mu : M \to \mathfrak{g}^*$, then the leaves of Δ_s are level sets $\mu^{-1}(\lambda)$, and Theorem 5.2 simply reproduces the usual Marsden–Weinstein quotient $\mu^{-1}(\lambda)/G_\lambda$.

If the symplectic G-action on M is free and proper, then ω induces a Poisson structure Π on M/G. We saw in Example 3.14 that the reduced manifolds fit into a singular foliation of M/G, which coincides with the symplectic foliation of Π. The reduction of $\mathcal{J}_\omega$ to each leaf can be obtained by the Dirac push-forward of ω to M/G, which is just Π, followed by the Dirac pull-back of Π to the leaf, which is the symplectic structure induced by Π on that leaf.

Next, we show that by allowing the projection $\pi : K \to TM$ to be injective, one can reduce a symplectic structure (type 0) to a generalized complex structure with nonzero type.

Example 5.6. Assume that X and Y are linearly independent symplectic vector fields generating a T^2-action on M. Assume further that $\omega(X, Y) = 0$ and consider the extended T^2-action on $TM \oplus T^*M$ defined by

$$\rho(\alpha_1) = X + \omega(Y); \qquad \rho(\alpha_2) = -Y + \omega(X),$$

where $\{\alpha_1, \alpha_2\}$ is the standard basis of $\mathfrak{t}^2 = \mathbb{R}^2$. It follows from $\omega(X, Y) = 0$ and the fact that the vector fields X and Y are symplectic that this is an extended action with isotropic K.

Since $\mathcal{J}_\omega K = K$, Theorem 5.2 implies that the quotient M/T^2 has an induced generalized complex structure. Note that

$$L \cap K_{\mathbb{C}}^{\perp} = \{Z - i\omega(Z) \colon Z \in \operatorname{Ann}(\omega(X) \wedge \omega(Y))\},$$

and it is simple to check that $X - i\omega(X) \in L \cap K_{\mathbb{C}}^{\perp}$ represents a nonzero element in $L_{\mathrm{red}} = ((L \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}})/K_{\mathbb{C}})/G$ which lies in the kernel of the projection $L_{\mathrm{red}} \to T(M/T^2)$. As a result, this reduced generalized complex structure has type 1.

One can find concrete examples illustrating this construction by considering symplectic manifolds which are T^2-principal bundles with Lagrangian fibres, such as $T^2 \times T^2$, or the Kodaira–Thurston manifold. In these cases, the reduced generalized complex structure determines a complex structure on the base 2-torus.

5.3. *Complex structures*

In this section we show how a complex manifold (M, I) may have different types of generalized complex reductions.

Example 5.7 *(Holomorphic quotient).* Let G be a complex Lie group acting holomorphically on (M, I), so that the induced infinitesimal map $\rho : \mathfrak{g} \to \Gamma(TM)$ is a holomorphic map. Since $K = \rho(\mathfrak{g}) \subset TM$, it is clear that K is isotropic and the reduced Courant algebroid is exact. Furthermore, as ρ is holomorphic, it follows that $\mathcal{J}_I K = K$. By Theorem 5.2, the complex structure descends to a generalized complex structure in the reduced manifold M/G. The reduced generalized complex structure is nothing but the quotient complex structure obtained from holomorphic quotient.

The previous example is a particular case of a more general fact: if (M, I) is a complex manifold, then any reduction of $\mathcal{J}_I$ by an extended action satisfying $\mathcal{J}_I K = K$ results in a generalized complex structure of complex type. Indeed, T^*M_{red} can be identified with

$$\frac{(K^\perp \cap T^*M + K \cap K^\perp)|_P}{(K \cap K^\perp)|_P} \Big/ G \subset E_{\mathrm{red}}$$

and using that $\mathcal{J}_I(T^*M) = T^*M$, one sees that $\mathcal{J}^{\mathrm{red}}(T^*M_{\mathrm{red}}) = T^*M_{\mathrm{red}}$, i.e., $\mathcal{J}^{\mathrm{red}}$ is of complex type. However, using Proposition 5.4, one can produce reductions of complex structures which are *not* of complex type.

Example 5.8. Consider $\mathbb{C}^2$ equipped with its standard holomorphic coordinates $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$, and let ρ be the extended $\mathbb{R}^2$-action on $\mathbb{C}^2$ defined by

$$\rho(\alpha_1) = \partial_{x_1} + dx_2, \qquad \rho(\alpha_2) = \partial_{y_2} + dy_1,$$

where $\{\alpha_1, \alpha_2\}$ is the standard basis for $\mathbb{R}^2$. Note that $K = \rho(\mathbb{R}^2)$ is isotropic, so the reduced Courant algebroid over $\mathbb{C}/\mathbb{R}^2$ is exact. Since the natural pairing between K and $\mathcal{J}_I K$ is nondegenerate, Proposition 5.4 implies that one can reduce $\mathcal{J}_I$ by this extended action. In this example, one computes

$$K_{\mathbb{C}}^\perp \cap L = \operatorname{span}\{\partial_{x_1} - i\partial_{x_2} - dy_1 + i\,dx_1, \partial_{y_1} - i\partial_{y_2} - dy_2 + i\,dx_2\}$$

and $K_{\mathbb{C}}^\perp \cap L \cap K_{\mathbb{C}} = \{0\}$. As a result, $L_{\mathrm{red}} \cong K_{\mathbb{C}}^\perp \cap L$. So $\pi : L_{\mathrm{red}} \to \mathbb{C}^2/\mathbb{R}^2$ is an injection, and $\mathcal{J}^{\mathrm{red}}$ has zero type, i.e., it is of symplectic type.

5.4. Extended Hamiltonian actions

In order to reduce a generalized complex structure $\mathcal{J}$ preserved by an extended action, we saw in Theorem 5.2 that a sufficient condition is the compatibility $\mathcal{J}K = K$. Natural examples where this condition holds arise as follows: one starts with an action generated by sections $v_i \in \Gamma(E)$, and then enlarges it to a new extended action generated by sections

$$\{v_i, \mathcal{J}v_j\}. \tag{31}$$

Examples where this construction works are the extended actions associated with symplectic and holomorphic actions: in the symplectic case (see Example 2.16), one starts with symplectic vector fields X_i and defines an extended action of the hemisemidirect Courant algebra, by adding new generators $\mathcal{J}_\omega(X_j) = \omega(X_j)$, which act as closed 1-forms; in the holomorphic case, one starts with an action generated by X_i preserving a complex structure I, and then forms the (trivially) extended action of the complexified Lie algebra, generated by $\{X_i, \mathcal{J}_I X_j\}$, where now $\mathcal{J}_I X_j = IX_j$ are new vector fields.

The "complexification" (31) does not always define an extended action, as we will see. However, in the case of a *Hamiltonian* action we show that it does produce an example of an extended action satisfying $\mathcal{J}K = K$.

It is familiar in the case of a complex manifold that a real vector field X preserves the complex structure I if and only if its $(1,0)$ component $X^{1,0} \in T_{1,0}M$ is a holomorphic vector field.

Therefore $IX = iX^{1,0} - iX^{0,1}$ also preserves the complex structure. In particular, if X generates an S^1 action then $\{X, IX\}$ defines a holomorphic $\mathbb{C}^*$ action on the complex manifold.

For generalized complex structures a similar phenomenon occurs, except that symmetries are governed by the differential complex $(\Omega^\bullet(L) = \Gamma(\wedge^\bullet L^*), d_L)$ associated to the complex Lie algebroid L defined by the $+i$-eigenbundle of $\mathcal{J}$.

Lemma 5.9. *A real section $v \in \Gamma(E)$ preserves the generalized complex structure $\mathcal{J}$ under the adjoint action if and only if $d_L v^{0,1} = 0$, where $v = v^{1,0} + v^{0,1} \in L \oplus \overline{L} = E \otimes \mathbb{C}$ and we use the inner product to identify $\overline{L} = L^*$.*

Proof. A real section $v \in \Gamma(E)$ preserves $\mathcal{J}$ if and only if $[v, \Gamma(L)] \subset \Gamma(L)$. Since L is maximal isotropic, it suffices to check that $\langle [v^{0,1}, w_1], w_2\rangle = 0$ for all $w_1, w_2 \in \Gamma(L)$. By definition of the Lie algebroid differential d_L, and using the basic properties of the Courant bracket, we have

$$\begin{aligned} d_L v^{0,1}(w_1, w_2) &= \pi(w_1)\langle v^{0,1}, w_2\rangle - \pi(w_2)\langle v^{0,1}, w_1\rangle - \langle v^{0,1}, [w_1, w_2]\rangle \\ &= 2\langle [v^{0,1}, w_2], w_1\rangle + \langle v^{0,1}, [w_1, w_2]\rangle + \pi(w_2)\langle v^{0,1}, w_1\rangle - \pi(w_1)\langle v^{0,1}, w_2\rangle \\ &= 2\langle [v^{0,1}, w_2], w_1\rangle - d_L v^{0,1}(w_1, w_2), \end{aligned}$$

so $d_L v^{0,1}(w_1, w_2) = \langle [v^{0,1}, w_2], w_1\rangle$, which immediately implies the result. □

We obtain the following exact sequence describing $\Gamma_{\mathcal{J}}(E)$, the space of sections of E preserving $\mathcal{J}$ under the adjoint action [9]:

$$C^\infty(M, \mathbb{C}) \xrightarrow{D} \Gamma_{\mathcal{J}}(E) \to H^1(L) \to 0,$$

where $D(f) = d_L f + \overline{d_L f} \in \Gamma(E)$, and the final term denotes the first Lie algebroid cohomology of L. The sections of E which lie in the image of D are called *Hamiltonian* symmetries [9], in direct analogy with the symplectic case. Note that for $f \in C^\infty(M, \mathbb{C})$ we have by definition

$$d_L f = \frac{1}{2}(df + i\mathcal{J}df),$$

so that the operator D may be expressed as

$$Df = d(\operatorname{Re} f) - \mathcal{J}d(\operatorname{Im} f).$$

Also note that the projection $\pi(Df) \in \Gamma(TM)$ lies in the projection $\pi(\mathcal{J}(T^*M))$ of the Dirac structure $\mathcal{J}(T^*M) \subset E$ and hence is tangent to the symplectic leaves of the Poisson structure induced by $\mathcal{J}$. This places a strong constraint on Hamiltonian symmetries which is familiar from the situation in Poisson geometry.

Example 5.10. In the symplectic case, a section $X + \xi \in \Gamma(TM \oplus T^*M)$ preserves $\mathcal{J}_\omega$ precisely when X is a symplectic vector field and $d\xi = 0$, whereas it is Hamiltonian if and only if X is Hamiltonian in the usual sense and ξ is exact. In the complex case, $X + \xi$ preserves $\mathcal{J}_I$ when $X^{1,0}$ is holomorphic and $\overline{\partial}\xi^{0,1} = 0$, whereas it is Hamiltonian if and only if $X = 0$ and $\xi = \overline{\partial} f + \partial \overline{f}$ for $f \in C^\infty(M, \mathbb{C})$.

We have the following immediate consequence of Lemma 5.9.

Corollary 5.11. *If $v \in \Gamma(E)$ preserves $\mathcal{J}$ then so does $\mathcal{J}v = iv^{1,0} - iv^{0,1}$.*

However, if the infinitesimal action of v integrates to an extended action on E, then this does not guarantee that $\mathcal{J}v$ also does, as we now show.

Example 5.12. Let $\mathfrak{h} = \mathfrak{a} = \mathbb{R}$ be a Courant algebra over the trivial Lie algebra $\mathfrak{g} = \{0\}$ and consider an action by covectors $\rho : \mathfrak{a} \to T^*M \subset TM \oplus T^*M$. In order that ρ define an extended action we need $\xi = \rho(1) \in \Omega^1_{\mathrm{cl}}(M)$. If M is endowed with a complex structure I, then the complexification of ρ satisfies $\rho_{\mathbb{C}}(i) = I^*\xi$, which is closed only if $d^c\xi = 0$.

While the "complexification" proposed in (31) may be obstructed because of the fact that $\mathcal{J}v$ may not define an extended action even if v does, we now show that if the given action is *Hamiltonian*, then it is equivalent, in the sense of Definition 2.10, to an action which can be extended so that $\mathcal{J}K = K$.

Theorem 5.13. *Let $\rho : \mathfrak{g} \to \Gamma(E)$ be a trivially extended, isotropic, Hamiltonian action on a generalized complex manifold, i.e. $\rho(a) = D(f_a)$ for a $\mathfrak{g}$-equivariant function $f : M \to \mathfrak{g}^*_{\mathbb{C}}$. Then the equivalent action $\tilde{\rho}(a) = \rho(a) - d(\mathrm{Re}\, f_a)$ may be extended to an action of the hemisemidirect Courant algebra $\mathfrak{g} \oplus \mathfrak{g}$, with moment map $\mathrm{Im}\, f$, and which satisfies the condition $\mathcal{J}K = K$.*

Proof. Since $\rho(a) = D(f_a) = d(\mathrm{Re}\, f_a) - \mathcal{J}d(\mathrm{Im}\, f_a)$, we see that

$$\mathcal{J}\tilde{\rho}(a) = d(\mathrm{Im}\, f_a),$$

which shows that the map $\rho' : \mathfrak{g} \oplus \mathfrak{g} \to \Gamma(E)$ given by

$$\rho' : (g, h) \mapsto \tilde{\rho}(g) + d(\mathrm{Im}\, f_h)$$

defines an extended action, as we saw in Proposition 2.17, and by construction satisfies $\mathcal{J}K = K$. □

Although this theorem concerns only Hamiltonian actions, which for generalized complex structures is increasingly restrictive as the type grows, we will use it to construct new examples of generalized Kähler structures (see Section 6). Also note that Examples 5.6, 5.7 and 5.8 are not Hamiltonian. We remark that the actions which are independently described by Lin and Tolman [20], as well as Hu [13], can be seen to be of this Hamiltonian type.

Finally, we provide a cohomological criterion which determines if a given action is Hamiltonian. If a trivially extended, isotropic action $\rho : \mathfrak{g} \to \Gamma(E)$ preserving $\mathcal{J}$ is given, then we may decompose $\rho(a) = Z_a + \zeta_a \in L \oplus \overline{L}$ for all $a \in \mathfrak{g}_{\mathbb{C}}$. Since this is a Courant morphism, we may then define an equivariant Cartan model for the differential complex $(\Omega^\bullet(L), d_L)$, by considering equivariant polynomial functions $\Phi : \mathfrak{g}_{\mathbb{C}} \to \Omega^\bullet(L)$, and equivariant derivative

$$(d_{\mathfrak{g}_{\mathbb{C}}}\Phi)(a) = d\big(\Phi(a)\big) - i_{Z_a}\Phi(a), \quad \forall a \in \mathfrak{g}_{\mathbb{C}}.$$

Since $\rho(a)$ preserves $\mathcal{J}$, we see that $\zeta_a \in \mathfrak{g}_{\mathbb{C}}^* \otimes \Omega^1(L)$ defines an equivariant closed 3-form. Supposing that $[\zeta_a] = 0$ in $H^3_{\mathfrak{g}_{\mathbb{C}}}(L)$, we then have

$$\zeta_a = d_{\mathfrak{g}_{\mathbb{C}}}(\varepsilon + h_a),$$

for $\varepsilon \in \Omega^2(L)$ an invariant d_L-closed form and $h_a \in \mathfrak{g}_{\mathbb{C}}^* \otimes \Omega^0(L)$ an equivariant function. Supposing further that $[\varepsilon] = 0$ in the invariant cohomology $H^2(L)^{\mathfrak{g}_{\mathbb{C}}}$, then $\varepsilon = d_L \eta$ for an invariant 1-form η, and

$$\zeta_a = d_{\mathfrak{g}_{\mathbb{C}}}(h_a + i_{Z_a}\eta),$$

implying that $\rho(a) = D(f_a)$ $\forall a \in \mathfrak{g}$, where $f_a = h_a + i_{z_a}\eta$. This provides the following result.

Proposition 5.14. *Let ρ be a trivially extended, isotropic action preserving a generalized complex structure. Then it is Hamiltonian if and only if the classes $[\zeta_a] \in H^3_{\mathfrak{g}_{\mathbb{C}}}(L)$ and $[\varepsilon] \in H^2(L)^{\mathfrak{g}_{\mathbb{C}}}$, defined above, vanish.*

6. Generalized Kähler reduction

A *generalized Kähler structure* [9] on an exact Courant algebroid E is a pair of commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ such that

$$\langle \mathcal{J}_1\mathcal{J}_2 v, v\rangle > 0 \quad \text{for all } v \in E.$$

The symmetric endomorphism $\mathcal{G} = \mathcal{J}_1\mathcal{J}_2$ therefore defines a positive-definite metric on E, called the *generalized Kähler metric*.

6.1. Reduction procedure

In this section we follow the standard treatment of Kähler reduction [10,18] and extend it to the generalized setting.

Theorem 6.1 *(Generalized Kähler reduction). Let E, ρ, and P be as in Theorem* 3.3*, with $\rho(\mathfrak{a}) = K$ isotropic along P. If the action preserves a generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ and $\mathcal{J}_1 K = K$ along P, Then $\mathcal{J}_1$ and $\mathcal{J}_2$ reduce to a generalized Kähler structure on E_{red}.*

Proof. Since K is isotropic, the reduced Courant algebroid is exact, and by Theorem 5.2, $\mathcal{J}_1$ descends to E_{red}. In order to show that $\mathcal{J}_2$ also descends, we will find an identification of E_{red} with a subbundle of $K^\perp$ which is invariant by $\mathcal{J}_2$.

Let $K^{\mathcal{G}}$ denote the orthogonal of K with respect to the metric $\mathcal{G}$. Since $\mathcal{J}_1 K = K$,

$$K^{\mathcal{G}} = (\mathcal{J}_2\mathcal{J}_1 K)^\perp = (\mathcal{J}_2 K)^\perp = \mathcal{J}_2 K^\perp \quad \text{over } P. \tag{32}$$

Since $K \subset K^\perp$ along P, we have the $\mathcal{G}$-orthogonal decomposition of $K^\perp$ as

$$K^\perp = K \oplus \left(K^{\mathcal{G}} \cap K^\perp\right) \quad \text{over } P.$$

It follows from (32) that $K^{\mathcal{G}} \cap K^{\perp}$ is $\mathcal{J}_2$-invariant. Using the natural identification

$$K^{\perp}/K \cong K^{\mathcal{G}} \cap K^{\perp} \quad \text{over } P$$

and the fact that $\mathcal{J}_2$ is G-invariant, we obtain after quotienting by G an induced orthogonal endomorphism $\mathcal{J}_2^{\mathrm{red}} : E_{\mathrm{red}} \to E_{\mathrm{red}}$ satisfying $(\mathcal{J}_2^{\mathrm{red}})^2 = -1$. It remains to check that $\mathcal{J}_2^{\mathrm{red}}$ is integrable.

In order to verify integrability, we first describe the $+i$-eigenbundle of $\mathcal{J}_2^{\mathrm{red}}$. Let L_2 be the $+i$-eigenbundle of $\mathcal{J}_2$. The $+i$-eigenbundle of $\mathcal{J}_2^{\mathrm{red}}$ is the image under the natural projection $p : K_{\mathbb{C}}^{\perp}|_P \to E_{\mathrm{red}} \otimes \mathbb{C}$ of $L_2 \cap (K_{\mathbb{C}}^{\perp} \cap K_{\mathbb{C}}^{\mathcal{G}})$. But since

$$L_2 \cap K_{\mathbb{C}}^{\perp} = \mathcal{J}_2\big(L_2 \cap K_{\mathbb{C}}^{\perp}\big) = L_2 \cap \mathcal{J}_2\big(K_{\mathbb{C}}^{\perp}\big) = L_2 \cap K_{\mathbb{C}}^{\mathcal{G}} \quad \text{over } P,$$

it follows that the $+i$-eigenbundle of $\mathcal{J}_2^{\mathrm{red}}$ is

$$p\big(L \cap \big(K_{\mathbb{C}}^{\perp} \cap K_{\mathbb{C}}^{\mathcal{G}}\big)\big) = p\big(L \cap K_{\mathbb{C}}^{\perp}\big) = (L_2)_{\mathrm{red}},$$

i.e., the reduction of the Dirac structure L_2. It follows that $(L_2)_{\mathrm{red}}$ is a smooth and maximal isotropic subbundle of $E_{\mathrm{red}} \otimes \mathbb{C}$, and by Theorem 4.2 we know that it is integrable. So $\mathcal{J}_2^{\mathrm{red}}$ is integrable.

Finally, we need to show that $(\mathcal{J}_1^{\mathrm{red}}, \mathcal{J}_2^{\mathrm{red}})$, where $\mathcal{J}_1^{\mathrm{red}}$ is the reduction of L_1, is a generalized Kähler pair in E_{red}. For that, we note that $K^{\mathcal{G}} \cap K^{\perp}$ is $\mathcal{J}_1$-invariant along P, since $\mathcal{J}_1(K^{\perp}) = K^{\perp}$ and $\mathcal{J}_1(K^{\mathcal{G}}) = K^{\mathcal{G}}$. So $\mathcal{J}_1$ induces an endomorphism of E_{red}, which coincides with the Dirac reduction $\mathcal{J}_1^{\mathrm{red}}$ since they have the same $+i$-eigenbundle: indeed,

$$\frac{L_1 \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}}}{K_{\mathbb{C}}} = \frac{(L_1 \cap K_{\mathbb{C}}) \oplus (L_1 \cap K_{\mathbb{C}}^{\mathcal{G}} \cap K_{\mathbb{C}}^{\perp}) + K_{\mathbb{C}}}{K_{\mathbb{C}}} \quad \text{over } P,$$

therefore, after quotienting by G, we get $(L_1)_{\mathrm{red}} = p(L_1 \cap K_{\mathbb{C}}^{\mathcal{G}} \cap K_{\mathbb{C}}^{\perp})$. The fact that $\mathcal{J}_1^{\mathrm{red}}$ and $\mathcal{J}_2^{\mathrm{red}}$ form a generalized Kähler pair is now a direct consequence of the fact that the restrictions of $\mathcal{J}_1$ and $\mathcal{J}_2$ to $K^{\mathcal{G}} \cap K^{\perp}$ commute and their product is positive definite. □

An important particular case of Theorem 6.1 is when the extended action admits a moment map.

Corollary 6.2. *Let $(\mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure preserved by an extended action admitting a moment map $\mu : M \to \mathfrak{h}^*$. Assume that the G-action on $\mu^{-1}(0)$ is free and proper. If $\mathcal{J}_1(K) = K$ over $\mu^{-1}(0)$, and the induced symmetric form $c_0 \in S^2\mathfrak{g}^*$ vanishes, then $\mathcal{J}_1$ and $\mathcal{J}_2$ can be reduced to M_{red} and define a generalized Kähler structure.*

This corollary follows from the fact that if c_0 vanishes, then both the isotropy action and the full action along $\mu^{-1}(0)$ are isotropic, i.e. $K \subset K^{\perp}$ on the level set. Of course these hypotheses are all fulfilled for a complexified Hamiltonian action as in Theorem 5.13. We now state the particular case when $\mathcal{J}_1$ is a symplectic structure since we use it in the next section.

Corollary 6.3. *Let $(\mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure on $E = TM \oplus T^*M$ with $H = 0$, such that $\mathcal{J}_1$ is an ordinary symplectic structure. Assume that there is a Hamiltonian action on $(M, \mathcal{J}_1)$, with moment map $\mu : M \to \mathfrak{g}^*$, and preserving $\mathcal{J}_2$. If the action of G on $\mu^{-1}(0)$ is free and proper, then the symplectic reduced space $M_{\mathrm{red}} = \mu^{-1}(0)/G$ carries a generalized Kähler structure given by $(\mathcal{J}_1^{\mathrm{red}}, \mathcal{J}_2^{\mathrm{red}})$.*

This result was independently obtained in [20], where it is used to produce many examples of generalized Kähler quotients. Also, when $\mathcal{J}_2$ is a complex structure, then $\mathcal{J}_2^{\mathrm{red}}$ is as well, and we recover the original Kähler reduction of [10,18].

Example 6.4 *(Symplectic cut).* Let $(\mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler manifold as in Corollary 6.3. Assume that there is a Hamiltonian S^1-action on M preserving $\mathcal{J}_2$, and let $f : M \to \mathbb{R}$ be its moment map. Consider $\mathbb{C}$ with its natural Kähler structure (ω, I), and equipped with the S^1-action $\theta \cdot z := e^{i\theta} z$. Then $N = M \times \mathbb{C}$ has a generalized Kähler structure $(\mathcal{J}_1', \mathcal{J}_2')$, where $\mathcal{J}_1'$ is the product symplectic structure and $\mathcal{J}_2' = \mathcal{J}_2 \times I$, and

$$\mu : N \to \mathbb{R}; \quad \mu(p, z) = f(p) + |z|^2$$

is a moment map for the diagonal S^1-action on N. This action preserves the generalized Kähler structure so, by Corollary 6.3, the symplectic quotient of N inherits a generalized Kähler structure.

6.2. *Examples of generalized Kähler structures on $\mathbb{C}P^2$*

Now we apply the results from the last section to produce new examples of generalized Kähler structure on $\mathbb{C}P^2$ with type change. The method consists of deforming the standard Kähler structure in $\mathbb{C}^3$ so that the deformed structure is still preserved by the circle action

$$e^{i\theta} : (z_1, z_2, z_3) \mapsto \left(e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3\right). \tag{33}$$

Then Corollary 6.3 implies that $\mathbb{C}P^2$, regarded as a symplectic reduction of $\mathbb{C}^3$, inherits a reduced generalized Kähler structure.

In the computations that follow, it will be convenient to use differential forms to describe a generalized complex structure $\mathcal{J}$ on a manifold M. So we recall from [9] that $\mathcal{J}$ is completely determined by its *canonical line bundle*, $C \subset \wedge^\bullet T^*_{\mathbb{C}} M$. This bundle is defined as the Clifford annihilator of L, the $+i$-eigenspace of $\mathcal{J}$. The fact that L is a Dirac structure of real index zero $(L \cap \overline{L} = \{0\})$ translates into properties for C: if φ is a nonvanishing local section of C, then:

- At each point, $\varphi = e^{B + i\omega} \wedge \Omega$, where B and ω are real 2-forms and Ω is a decomposable complex k-form;
- There is a local section $X + \xi \in \Gamma(TM \oplus T^*M)$ such that

$$d\varphi = (X + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi;$$

- If σ is the linear map which acts on k-forms by $\sigma(a) = (-1)^{\frac{k(k-1)}{2}} a$, then the Mukai pairing $(\varphi, \overline{\varphi})$ must be nonzero, where

$$(\varphi, \overline{\varphi}) := \left(\varphi \wedge \sigma(\overline{\varphi})\right)_{\mathrm{top}}.$$

The subscript *top* indicates a projection to the volume form component.

We begin with the standard Kähler structure on $(\mathbb{C}^3, \mathcal{J}_\omega, \mathcal{J}_I)$, defined by the following differential forms:

$$\Omega = dz_0 \wedge dz_1 \wedge dz_2,$$
$$\omega = \frac{i}{2}(dz_0 \wedge d\bar{z}_0 + dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2).$$

As explained in [9], it is possible to deform this Kähler structure as a generalized Kähler structure in such a way that ω is unchanged whereas the complex structure Ω becomes a generalized complex structure of generic type 1. To achieve this, we must select a deformation $\varepsilon \in \Gamma(L_+^* \otimes L_-^*)$, where

$$L_\pm^* = \{X \pm i\omega(X)\colon\ X \in TM^{1,0}\},$$

which satisfies the Maurer–Cartan equation $\bar{\partial}\varepsilon + \frac{1}{2}[\varepsilon, \varepsilon] = 0$. Then in regions where ε does not invalidate the open condition that $e^\varepsilon \Omega$ be of real index zero, $(e^\varepsilon \Omega, e^{i\omega})$ will be a generalized Kähler pair.

Example 6.5. In this example we deform the structure in $\mathbb{C}^3$ so that the reduced structure in $\mathbb{C}P^2$ has type change along a triple line. A similar deformation and quotient has been considered independently by Lin and Tolman in [20], where they also consider a variety of other examples. A generalized Kähler structure on $\mathbb{C}P^2$ with type change along a triple line was also recently constructed by Hitchin [12] using a different method.

The deformation. We select the decomposable element

$$\varepsilon = \frac{1}{2}z_0^2\left(\partial_1 + \frac{1}{2}\,dz_1\right) \wedge \left(\partial_2 - \frac{1}{2}\,dz_2\right),$$

whose bivector component $\frac{1}{2}z_0^2\partial_1 \wedge \partial_2$ is a quadratic holomorphic Poisson structure. The projectivization of this structure is a Poisson structure on $\mathbb{C}P^2$ vanishing to order 3 along the line $z_0 = 0$. The deformed complex structure in $\mathbb{C}^3$ can be written explicitly (we omit the wedge symbol):

$$\begin{aligned}
\varphi &= e^\varepsilon dz_0\,dz_1\,dz_2 = (1+\varepsilon)\,dz_0\,dz_1\,dz_2 \\
&= dz_0\,dz_1\,dz_2 - \frac{1}{2}z_0^2\,dz_0 - \frac{1}{4}z_0^2\,dz_0\,dz_2\,d\bar{z}_2 + \frac{1}{4}z_0^2\,dz_0\,dz_1\,d\bar{z}_1 + \frac{1}{8}z_0^2\,dz_0\,dz_1\,dz_2\,d\bar{z}_2\,d\bar{z}_1 \\
&= -\frac{1}{2}z_0^2\,dz_0 \exp\left(-\frac{2}{z_0^2}\,dz_1\,dz_2 + \frac{1}{2}(dz_2\,d\bar{z}_2 - dz_1\,d\bar{z}_1)\right). \qquad (34)
\end{aligned}$$

Let $\zeta = -\frac{1}{2}z_0^2\,dz_0$ and $b + i\sigma = -\frac{2}{z_0^2}\,dz_1\,dz_2 + \frac{1}{2}(dz_2\,d\bar{z}_2 - dz_1\,d\bar{z}_1)$. Then the pure differential form φ is of real index zero as long as the Mukai pairing of φ with its complex conjugate satisfies

$$(\varphi, \bar{\varphi}) = \sigma^2 \wedge \zeta \wedge \bar{\zeta} \neq 0.$$

Calculating this quantity, we obtain:

$$\sigma^2 \wedge \zeta \wedge \bar{\zeta} = \frac{1}{2}\big(4 - |z_0|^4\big)\, dz_0\, dz_1\, dz_2\, d\bar{z}_0\, d\bar{z}_1\, d\bar{z}_2,$$

proving that $(\varphi, e^{i\omega})$ defines a generalized Kähler structure in $\mathbb{C}^3$ away from the cylinder $|z_0| = \sqrt{2}$.

The reduction. Notice that the line generated by φ, and hence the generalized complex structure it defines, is invariant by the S^1-action given by (33). Hence, by Corollary 6.3, the symplectic reduction of $\mathbb{C}^3$ will have a reduced generalized Kähler structure induced by the deformed structure above. We spend the rest of this example describing this structure. The particular reduction we wish to calculate is the quotient of the unit sphere $\sum_i z_i \bar{z}_i = 1$ by the S^1-action given by (33).

We begin with the generalized complex structure φ given by Eq. (34). The induced Dirac structure on the reduced Courant algebroid may be calculated by pulling back to the unit sphere in $\mathbb{C}^3$ and pushing forward to the quotient. The latter operation on differential forms may be expressed simply as interior product with ∂_θ, the generator of the circle action

$$\partial_\theta = i(z_0\partial_0 - \bar{z}_0\bar{\partial}_0 + z_1\partial_1 - \bar{z}_1\bar{\partial}_1 + z_2\partial_2 - \bar{z}_2\bar{\partial}_2),$$

and this commutes with pull-back to the sphere. So let us first take interior product:

$$\begin{aligned} i_{\partial_\theta}\varphi &= (i_{\partial_\theta}\zeta)\exp\left(\frac{-\zeta \wedge i_{\partial_\theta}(b + i\sigma)}{i_{\partial_\theta}\zeta} + b + i\sigma\right) \\ &= -\frac{i}{2} z_0^3 \exp\left(-\frac{dz_0}{z_0}\left(\frac{2(z_2\,dz_1 - z_1\,dz_2)}{z_0^2} + \frac{z_2\,d\bar{z}_2 + \bar{z}_2\,dz_2 - z_1\,d\bar{z}_1 - \bar{z}_1\,dz_1}{2}\right)\right. \\ &\qquad \left. - \frac{2\,dz_1\,dz_2}{z_0^2} + \frac{dz_2\,d\bar{z}_2 - dz_1\,d\bar{z}_1}{2}\right). \end{aligned}$$

Now we pull back to S^5 by imposing $1 = R^2 = \sum_i z_i\bar{z}_i$ and obtain a homogeneous differential form after rescaling:

$$\begin{aligned} \tilde{\varphi} = \exp\left(-\frac{dz_0}{z_0}\left(\frac{2(z_2\,dz_1 - z_1\,dz_2)}{z_0^2} + \frac{z_2\,d\bar{z}_2 + \bar{z}_2\,dz_2 - z_1\,d\bar{z}_1 - \bar{z}_1\,dz_1}{2R^2}\right)\right. \\ \left. - \frac{2\,dz_1\,dz_2}{z_0^2} + \frac{dz_2\,d\bar{z}_2 - dz_1\,d\bar{z}_1}{2R^2}\right). \end{aligned}$$

The holomorphic Euler vector field is $\mathsf{e} = \sum_i z_i\partial_i$ and $\partial_\theta = i(\mathsf{e} - \bar{\mathsf{e}})$. The radial vector field is $\partial_r = \mathsf{e} + \bar{\mathsf{e}}$. In order to be the pull-back of a form on $\mathbb{C}P^2$, a differential form α on $\mathbb{C}^3$ must satisfy $\mathcal{L}_{\mathsf{e}}\alpha = \mathcal{L}_{\bar{\mathsf{e}}}\alpha = i_{\mathsf{e}}\alpha = i_{\bar{\mathsf{e}}}\alpha = 0$. We have already ensured that $\mathcal{L}_{\mathsf{e}}\tilde{\varphi} = \mathcal{L}_{\bar{\mathsf{e}}}\tilde{\varphi} = 0$ and $i_{\mathsf{e}-\bar{\mathsf{e}}}\tilde{\varphi} = 0$, so now we may add a multiple of dR to ensure $i_{\mathsf{e}+\bar{\mathsf{e}}}\tilde{\varphi} = 0$. Since dR vanishes on the sphere, this is a trivial modification.

Recall that $i_{\mathsf{e}+\bar{\mathsf{e}}}\frac{dR}{R} = 1$, so we shall subtract

$$\frac{dR}{R} \wedge i_{\mathsf{e}+\bar{\mathsf{e}}}\tilde{\varphi} = \frac{dR}{R}\left(\frac{dz_0}{z_0}\left(\frac{z_2\bar{z}_2 - z_1\bar{z}_1}{R^2}\right) + \frac{\bar{z}_1\,dz_1 - \bar{z}_2\,dz_2}{R^2}\right)\tilde{\varphi}.$$

Finally we get a manifestly projective representative for the generator of the canonical bundle:

$$\varphi_B = \exp\biggl(-\frac{dz_0}{z_0}\biggl(\frac{2(z_2\,dz_1 - z_1\,dz_2)}{z_0^2} + \frac{z_2\,d\bar{z}_2 + \bar{z}_2\,dz_2 - z_1\,d\bar{z}_1 - \bar{z}_1\,dz_1}{2R^2}\biggr) - \frac{2\,dz_1\,dz_2}{z_0^2} + \frac{dz_2\,d\bar{z}_2 - dz_1\,d\bar{z}_1}{2R^2} - \frac{dR}{R}\biggl(\frac{dz_0}{z_0}\biggl(\frac{z_2\bar{z}_2 - z_1\bar{z}_1}{R^2}\biggr) + \frac{\bar{z}_1\,dz_1 - \bar{z}_2\,dz_2}{R^2}\biggr)\biggr).$$

This differential form is closed, but blows up along the type change locus, where one can see by rescaling that it defines a complex structure. This generalized complex structure, together with the Fubini–Study symplectic structure, forms a generalized Kähler structure on $\mathbb{C}P^2$.

It may be of interest to express this generalized Kähler structure in affine coordinates (z_1, z_2) where $z_0 = 1$. Then the type change locus is the line at infinity. Define $r^2 = z_1\bar{z}_1 + z_2\bar{z}_2$:

$$\varphi_B = \exp\biggl(-2\,dz_1\,dz_2 + \frac{dz_2\,d\bar{z}_2 - dz_1\,d\bar{z}_1}{2(1+r^2)} - \frac{1}{2}\frac{d(r^2)(\bar{z}_1\,dz_1 - \bar{z}_2\,dz_2)}{(1+r^2)^2}\biggr).$$

The form defining the Fubini–Study symplectic form in these coordinates is, as usual:

$$\varphi_A = \exp\biggl(-\frac{1}{2}\frac{(1+r^2)(dz_1\,d\bar{z}_1 + dz_2\,d\bar{z}_2) - (\bar{z}_1\,dz_1 + \bar{z}_2\,dz_2)(z_1\,d\bar{z}_1 + z_2\,d\bar{z}_2)}{(1+r^2)^2}\biggr).$$

An important constituent of a generalized Kähler structure is its associated bi-Hermitian metric; this can be derived from the above forms as follows. Define real 2-forms ω_1, ω_2, b such that $\varphi_A = e^{i\omega_1}$ and $\varphi_B = e^{b+i\omega_2}$. Then the bi-Hermitian metric g is simply

$$g = -\omega_2 b^{-1}\omega_1.$$

Example 6.6. To demonstrate the versatility of the quotient construction we now construct a generalized Kähler structure on $\mathbb{C}P^2$ with type change along a slightly more general cubic: the union of three distinct lines forming a triangle. We postpone the discussion of the general cubic curve to a future paper.

The deformation. In this example we select a deformation ε given by the following decomposable section of $L_+^* \otimes L_-^*$:

$$\varepsilon = \frac{1}{2}\biggl(z_0\biggl(\partial_1 + \frac{1}{2}\,d\bar{z}_1\biggr) + z_1\biggl(\partial_2 + \frac{1}{2}\,d\bar{z}_2\biggr) + z_2\biggl(\partial_0 + \frac{1}{2}\,d\bar{z}_0\biggr)\biggr) \wedge \biggl(z_0\biggl(\partial_2 - \frac{1}{2}\,d\bar{z}_2\biggr) + z_1\biggl(\partial_0 - \frac{1}{2}\,d\bar{z}_0\biggr) + z_2\biggl(\partial_1 - \frac{1}{2}\,d\bar{z}_1\biggr)\biggr)$$

whose bivector component $\beta = (z_0^2 - z_1z_2)\partial_1\partial_2 + (z_1^2 - z_2z_0)\partial_2\partial_0 + (z_2^2 - z_0z_1)\partial_0\partial_1$ is a quadratic holomorphic Poisson structure on $\mathbb{C}^3$. This induces a Poisson structure on $\mathbb{C}P^2$ vanishing on the zero set of the following cubic polynomial:

$$\begin{aligned} \mathrm{e} \wedge \beta &= \bigl(z_0^3 + z_1^3 + z_2^3 - 3z_0z_1z_2\bigr)\partial_0\partial_1\partial_2 \\ &= (z_0 + z_1 + z_2)\bigl(z_0 + \lambda z_1 + \lambda^2 z_2\bigr)\bigl(z_0 + \lambda^2 z_1 + \lambda z_2\bigr)\partial_0\partial_1\partial_2, \end{aligned}$$

where $\mathrm{e} = \sum z_i \partial_i$ is the holomorphic Euler vector field and λ is a third root of unity. We see that the vanishing set of this Fermat cubic is the union of three distinct lines in the plane which intersect at the points $\{[1:1:1], [1:\lambda:\lambda^2], [1:\lambda^2:\lambda]\}$.

The deformed complex structure can be written explicitly:

$$\begin{aligned}\varphi &= e^{\varepsilon}\, dz_0\, dz_1\, dz_2 = (1+\varepsilon)\, dz_0\, dz_1\, dz_2\\ &= \left(\frac{1}{2}\bigl(-z_0^2+z_1z_2\bigr)\, dz_0 + \text{c.p.}\right)\exp\left(-\frac{1}{2}\frac{z_1^2+z_0z_2}{-z_2^2+z_0z_1}\, dz_1\, d\bar{z}_2 + \frac{1}{2}\frac{z_0^2+z_1z_2}{-z_2^2+z_0z_1}\, dz_0\, d\bar{z}_2 + \text{c.p.}\right),\end{aligned}$$

where "c.p." denotes cyclic permutations of $\{0, 1, 2\}$. The pure differential form φ is of real index zero as long as it has nonvanishing Mukai pairing with its complex conjugate:

$$\langle \varphi, \overline{\varphi} \rangle = \left(\frac{R^4}{4} - 1\right) dz_0\, dz_1\, dz_2\, d\bar{z}_0\, d\bar{z}_1\, d\bar{z}_2,$$

where $R^2 = |z_0|^2 + |z_1|^2 + |z_2|^2$. The generalized almost complex structure determined by φ on the ball of radius $\sqrt{2}$ is not integrable, however, since

$$d\varphi = \frac{1}{2}(z_0\, d\bar{z}_0 + z_1\, d\bar{z}_1 + z_2\, d\bar{z}_2)\, dz_0\, dz_1\, dz_2.$$

Nonetheless, when pulled back to the unit sphere in $\mathbb{C}^3$ this derivative vanishes, and hence we may proceed as before, quotienting by the S^1-action (33), as we do next.

The reduction. We begin with the generalized complex structure φ:

$$\varphi = \left(\frac{1}{2}\bigl(-z_0^2+z_1z_2\bigr)\, dz_0 + \text{c.p.}\right)\exp\left(-\frac{1}{2}\frac{z_1^2+z_0z_2}{-z_2^2+z_0z_1}\, dz_1\, d\bar{z}_2 + \frac{1}{2}\frac{z_0^2+z_1z_2}{-z_2^2+z_0z_1}\, dz_0\, d\bar{z}_2 + \text{c.p.}\right).$$

As in Example 6.5, we calculate the interior product by ∂_θ:

$$\begin{aligned} i_{\partial_\theta}\varphi = &-\frac{i(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)}{2}\\ &\times \exp\left(\frac{(z_2|z_0|^2 + z_2|z_1|^2 + z_0^2\bar{z}_1 + 2z_0z_1\bar{z}_2 + z_1^2\bar{z}_0)\, dz_0\, dz_1 + (z_1^3 - z_2^3)\, dz_0\, d\bar{z}_0}{2(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)}\right.\\ &\left. + \frac{(z_0^2z_1 - 2z_0z_2^2 + z_1^2z_2)\, dz_0\, d\bar{z}_1 - (z_0^2z_2 - 2z_0z_1^2 + z_1z_2^2)\, dz_0\, d\bar{z}_2}{2(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)} + \text{c.p.}\right).\end{aligned}$$

Now we pull back to S^5 by imposing $1 = R^2 = \sum_i z_i\bar{z}_i$ and obtain a homogeneous differential form after rescaling:

$$\begin{aligned}\tilde{\varphi} = &\exp\left(\frac{(z_2|z_0|^2 + z_2|z_1|^2 + z_0^2\bar{z}_1 + 2z_0z_1\bar{z}_2 + z_1^2\bar{z}_0)\, dz_0\, dz_1 + (z_1^3 - z_2^3)\, dz_0\, d\bar{z}_0}{2R^2(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)}\right.\\ &\left. + \frac{(z_0^2z_1 - 2z_0z_2^2 + z_1^2z_2)\, dz_0\, d\bar{z}_1 - (z_0^2z_2 - 2z_0z_1^2 + z_1z_2^2)\, dz_0\, d\bar{z}_2}{2R^2(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)} + \text{c.p.}\right).\end{aligned}$$

As in the previous example, we subtract from this the quantity $\frac{dR}{R} \wedge i_{e+\bar{e}}\tilde{\varphi}$, obtaining finally a manifestly projective representative for the generator for the canonical bundle:

$$\varphi_B = \exp\Bigg(\frac{((z_1^3 - z_2^3 - z_0^3 - z_0z_1z_2)|z_1|^2 - (z_2^3 - z_0^3 - z_1^3 - z_0z_1z_2)|z_2|^2 + 2z_0^2(z_2^2\bar{z}_1 - z_1^2\bar{z}_2))\,dz_0\,d\bar{z}_0}{2R^2(z_0^3 + z_1^3 + z_2^3 - 3z_0z_1z_2)}$$
$$+ \frac{(z_1\bar{z}_0(z_0^3 - z_1^3 + z_2^3 + z_0z_1z_2) - 2z_0\bar{z}_2(z_1^3 + z_2^3 - z_0z_1z_2) - 2|z_0|^2z_0z_2^2 + 2|z_2|^2z_2z_1^2)\,dz_0\,d\bar{z}_1}{2R^2(z_0^3 + z_1^3 + z_2^3 - 3z_0z_1z_2)}$$
$$+ \frac{(z_2\bar{z}_0(-z_0^3 - z_1^3 + z_2^3 - z_0z_1z_2) + 2z_0\bar{z}_1(z_1^3 + z_2^3 - z_0z_1z_2) + 2|z_0|^2z_0z_1^2 - 2|z_1|^2z_1z_2^2)\,dz_0\,d\bar{z}_2}{2R^2(z_0^3 + z_1^3 + z_2^3 - 3z_0z_1z_2)}$$
$$+ \frac{(z_2(|z_0|^2|z_1|^2 + z_0^2\bar{z}_1\bar{z}_2 + \bar{z}_0^2z_1z_2 + \text{c.p.}))\,dz_0\,dz_1}{2R^2(z_0^3 + z_1^3 + z_2^3 - 3z_0z_1z_2)} + \text{c.p.}\Bigg).$$

This differential form is closed, but blows up along the three distinct lines of the type change locus, where one can verify by rescaling that it defines a complex structure. This generalized complex structure, together with the Fubini–Study symplectic structure, forms a generalized Kähler structure on $\mathbb{C}P^2$.

In affine coordinates (z_1, z_2) for $\mathbb{C}P^2$, the type change locus consists of three lines intersecting at $\{(1,1), (\lambda, \lambda^2), (\lambda^2, \lambda)\}$. Define $r^2 = z_1\bar{z}_1 + z_2\bar{z}_2$. We may now write φ_B in these coordinates:

$$\varphi_B = \exp\Bigg(\frac{((z_2^3 - 1 - z_1^3 - z_1z_2)|z_2|^2 - (1 - z_1^3 - z_2^3 - z_1z_2) + 2z_1^2(\bar{z}_2 - z_2^2))\,dz_1\,d\bar{z}_1}{2(1+r^2)(1 + z_1^3 + z_2^3 - 3z_1z_2)}$$
$$+ \frac{((1 - z_1^3 - z_2^3 - z_1z_2) - (z_1^3 - z_2^3 - 1 - z_1z_2)|z_1|^2 + 2z_2^2(z_1^2 - \bar{z}_1))\,dz_2\,d\bar{z}_2}{2(1+r^2)(1 + z_1^3 + z_2^3 - 3z_1z_2)}$$
$$+ \frac{(z_2\bar{z}_1(z_1^3 - z_2^3 + 1 + z_1z_2) - 2z_1(z_2^3 + 1 - z_1z_2) - 2|z_1|^2z_1 + 2z_2^2)\,dz_1\,d\bar{z}_2}{2(1+r^2)(1 + z_1^3 + z_2^3 - 3z_1z_2)}$$
$$+ \frac{(z_1\bar{z}_2(-z_2^3 - 1 + z_1^3 - z_1z_2) + 2z_2(1 + z_1^3 - z_1z_2) + 2|z_2|^2z_2 - 2z_1^2)\,dz_2\,d\bar{z}_1}{2(1+r^2)(1 + z_1^3 + z_2^3 - 3z_1z_2)}$$
$$+ \frac{(|z_1|^2|z_2|^2 + z_1^2\bar{z}_2 + \bar{z}_1^2z_2 + z_1(|z_2|^2 + z_2^2\bar{z}_1 + \bar{z}_2^2z_1) + z_2(|z_1|^2 + \bar{z}_1\bar{z}_2 + z_1z_2))\,dz_1\,dz_2}{2(1+r^2)(1 + z_1^3 + z_2^3 - 3z_1z_2)}\Bigg).$$

This form, together with the Fubini–Study symplectic structure

$$\varphi_A = \exp\bigg(-\frac{1}{2}\frac{(1+r^2)(dz_1\,d\bar{z}_1 + dz_2\,d\bar{z}_2) - (\bar{z}_1\,dz_1 + \bar{z}_2\,dz_2)(z_1\,d\bar{z}_1 + z_2\,d\bar{z}_2)}{(1+r^2)^2}\bigg),$$

defines explicitly a generalized Kähler structure on $\mathbb{C}P^2$ with type change along a triangle as described above.

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