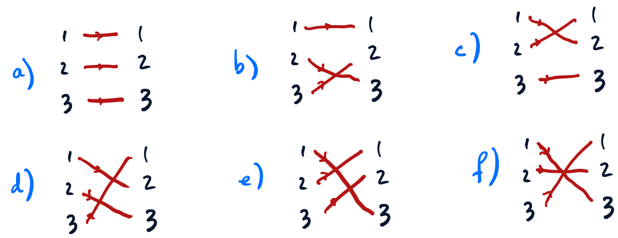
Inverses

The <u>inverse</u> of a map $f: X \to Y$ is a map $g: Y \to X$ such that $g \circ f = I_X$ <u>and</u> $f \circ g = I_Y$

If f has an inverse, we usually denote it by f^{-1}

Q.: Which of these maps has an inverse, and what is it?



Q.: Give an example of a non-invertible map

Q.: Let $f: X \to Y$ and $g: Y \to X$ be such that $g \circ f = I_X$.
Does it follow that $f \circ g = I_Y$? Why?

Q.: Let $f: X \to Y$ and $g: Y \to X$ be s.t. $f \circ g = I_Y$.
Under what condition on f could we prove $g \circ f = I_X$?

Q.: Prove that f is invertible if and only if it is a bijection.

The bijections from a set X to itself are very special: they form a group (a "permutation" group if X finite) and are therefore sometimes called "Symmetries" of X.

$$S_X = (\underbrace{\mathrm{Bij}(X,X)}_{\text{set of bijections } X \circlearrowleft}, \overset{\text{composition}}{\circ}, \underbrace{I_X}_{\text{Identity}})$$

The symmetry group of X.

How to construct sets

Subsets: Given a set Y, a subset $X \subseteq Y$ is a set comprised of some of the objects in Y. That is, every element of X is also an element of Y.

we can think of X as the set of elements of Y satisfying some constraints:

$$X = \{ n \in \mathbb{Z} : n \text{ is odd and } 1 \leq n \leq 6 \}$$

$$X = \{1, 3, 5\}$$ a set of three elements.

note that if $f : Y \to Z$ is a map and $X \subseteq Y$

If $f : Y \to Z$ then the restriction of f to $X \subseteq Y$ is the map

$$f|_X : X \to Z$$
$$x \longmapsto f(x)$$

You should be familiar with the basic operations on subsets: if X_1, X_2 are subsets of Y then

- $X_1 \cup X_2 = \{ x \in Y : x \in X_1 \text{ or } x \in X_2 \}$
  UNION
- $X_1 \cap X_2 = \{ x \in Y : x \in X_1 \text{ and } x \in X_2 \}$
  INTERSECTION

using these we can create many sets.

The **Power Set** $\mathcal{P}(X)$ of X is the set of all subsets of X.

Disjoint subsets: Subsets X_1, X_2 of Y are **disjoint** when $X_1 \cap X_2 = \phi$.

Partitions

If $X_1, \dots, X_k$ are pairwise disjoint nonempty subsets of Y and $Y = X_1 \cup \dots \cup X_k$, we say $\{X_1, \dots, X_k\}$ is a **Partition** of Y

Q.: List all partitions of $\{1,2,3,4\}$.

Building a map from pieces

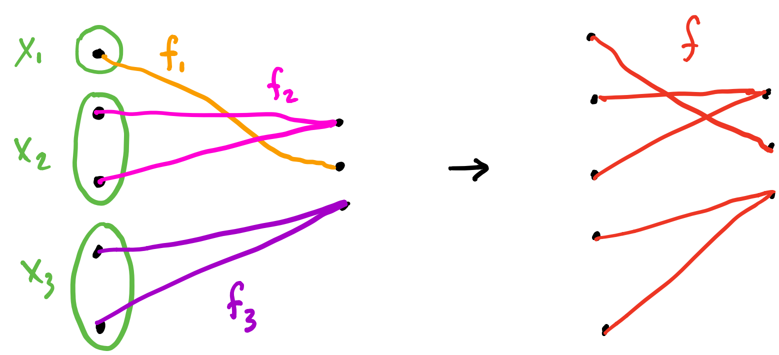
Let $\{X_1, \dots, X_k\}$ be a partition of Y, and let $f_i : X_i \to Z$ be a map for each $i \in \{1, \dots, k\}$.

then we can define a map $f : Y \longrightarrow Z$ as follows

For each $y \in Y$, there is a unique X_i containing it.

Then define $f(y) = f_i(y)$.

$$f = \bigcup_{i=1}^{k} f_i$$



Preimages

Fix a map $f: Y \to Z$. The Image (or range) of f is $f(Y) \subseteq Z$, where

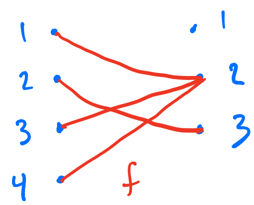
$$f(Y) = \{ f(y) : y \in Y \}.$$

There is also the idea of preimage (or inverse image)

For any $z \in Z$, its preimage is

$$f^{-1}(z) = \{ y \in Y : f(y) = z \}$$

Ex.:



- $\mathrm{Im}(f) = \{2,3\}$

$f^{-1}(1) = \phi$

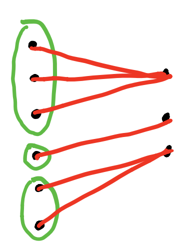
$f^{-1}(2) = \{1,3,4\}$.

$f^{-1}(3) = \{2\}$

Theorem: Any map $f: Y \to Z$ determines

- Image subset: $\mathrm{Im}(f) \subseteq Z$
- Partition of Y: $\{ f^{-1}(z) : z \in \mathrm{Im}(f) \}$.
  (labeled by $\mathrm{Im}(f)$)

and these data suffice to reconstruct the map: we simply let $f_z : f^{-1}(z) \to Z$ be constant, with value z.



The standard set of n elements

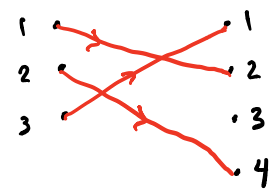
Fix $n \in \{1, 2, \cdots\}$ a natural number

The standard set of size n is $B_n = \{1, 2, \dots, n\}$

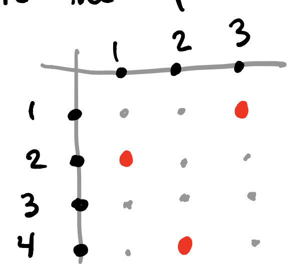
Q.: How many maps are there $B_m \to B_n$?

Instead of writing a map $f: B_3 \to B_4$ as a diagram

f :

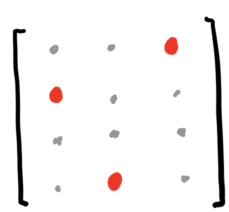


we could capture the information by drawing the graph



$\mathrm{Graph}(f) = \{(1,2), (2,4), (3,1)\}$

In other words, f can be described as a 4×3 grid



or

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where entries are either in the graph or not.

Equivalently, it is a 4×3 grid of 0's and 1's where each column has exactly one 1.