# Notes on Matrices

Let $V, W$ be finite-dimensional vector spaces over a field $\mathbb{F}$. Given a linear map $T : V \longrightarrow W$, we may decide to use a matrix to describe the map. To do this, we follow a certain convention. The purpose of these notes is to explain the convention.

## Coordinates of a vector as a $n \times 1$ matrix

Choose a basis $\beta = (v_1, \ldots, v_n)$ for $V$ Then any vector $x \in V$ can be written uniquely as $x = \sum_{i=1}^n x_i v_i$. This allows us to describe $x$ by giving its coordinates $(x_1, \ldots, x_n) \in \mathbb{F}^n$. We define the matrix of $x$ to be

$$_\beta x := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

## Matrix of a linear map

The linear map $T : V \longrightarrow W$ is completely determined by its values on the basis $\beta$. If we choose a basis $\gamma = (w_1, \ldots, w_k)$ for $W$, then the value of $T$ on the $j^{th}$ basis element $v_j$ of $\beta$ is

$$T(v_j) = \sum_{i=1}^{k} a_{ij} w_i. \tag{1}$$

This defines a $k \times n$ array of numbers $a_{ij} \in \mathbb{F}$ ($i$ indicates the row, $j$ indicates the column), which is defined to be the matrix of $T$ with respect to $\beta, \gamma$:

$$_\gamma T_\beta := \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix}$$

Note that with this definition, the coordinates of $Tv_j$ appear as the $j^{th}$ column of the matrix. Instead of writing $_\gamma T_\beta$ you could write $M(T, \beta, \gamma)$; ours is just a shorthand notation which helps to remember that $\beta$ is the basis for the domain of $T$ whereas $\gamma$ is the basis for the codomain. Note also that a vector $x \in V$ defines a map $\mathbb{F} \to V$ defined by $1 \mapsto x$. If we call this map $x$, then we can interpret $_\beta x$ as the matrix of the linear map $x$ written in the basis $\beta$ for the codomain and in the standard basis for the domain $\mathbb{F}$.

## Applying a matrix to a vector

A $k \times n$ matrix can be "multiplied" by or "applied" to a $n \times 1$ matrix to yield a $k \times 1$ matrix. By definition, we set

$$({}_\gamma T_\beta)({}_\beta x) := {}_\gamma(Tx),$$

in other words, the matrix of $T$ applied to the matrix of $x$ gives the matrix of $Tx$. To compute this, we use the fact

$$Tx = \sum_{j=1}^{n} x_j T(v_j) = \sum_{i=1}^{k}(\sum_{j=1}^{n} a_{ij}x_j)w_i.$$

Therefore, the $i^{th}$ entry in the $k \times 1$ matrix $({}_\gamma T_\beta)({}_\beta x)$ is $\sum_{j=1}^{n} a_{ij}x_j$.

## Composition as matrix multiplication

Let $U$ be another vector space, with basis $\alpha = (u_1, \ldots, u_m)$. If $S : U \longrightarrow V$ and $T : V \longrightarrow W$ are linear maps, then they can be composed to give $TS : U \longrightarrow W$. This implies that we should be able to "multiply" the $k \times n$ matrix of $T$ by the $n \times m$ matrix of $S$ to give the $k \times m$ matrix of $TS$. We define the matrix product

$$({}_\gamma T_\beta)({}_\beta S_\alpha) := {}_\gamma TS_\alpha.$$

Suppose that the matrix ${}_\beta S_\alpha$ has entries $b_{ij}$defined by $Su_j = \sum_{i=1}^{n} b_{ij}v_i$. Then we can compute the matrix of the composition above, using:

$$TS(u_j) = T(\sum_p b_{pj}v_p) = \sum_{i=1}^{k}\left(\sum_{p=1}^{n} a_{ip}b_{pj}\right) w_i.$$

Therefore, the $ij^{th}$ entry in the $k \times m$ matrix $({}_\gamma T_\beta)({}_\beta S_\alpha)$ is $\sum_{p=1}^{n} a_{ip}b_{pj}$, which is the usual formula given for the $ij^{th}$ entry of a matrix product (the dot product of the $i^{th}$ row of $T$ with the $j^{th}$ column of $S$). It is obvious that matrix multiplication is associative, since it is defined using composition of linear maps, and composition of maps is always associative.

## Change of basis

Suppose you know the matrix of $T : V \longrightarrow W$ with respect to bases $\beta, \gamma$ for $V, W$, and someone hands you new bases $\beta', \gamma'$. How can you find the matrix of $T$ in the new basis? We can use the simple fact that the identity maps $I_V$ and $I_W$ can be composed with $T$ without changing anything, i.e.

$$T = I_W T I_V.$$

But then we know from the definition of matrix multiplication that

$${}_{\gamma'}T_{\beta'} = {}_{\gamma'}(ITI)_{\beta'} = ({}_{\gamma'}I_\gamma)({}_\gamma T_\beta)({}_\beta I_{\beta'}).$$

The matrix $P = {}_\beta I_{\beta'}$ is usually called the "change of basis matrix" and its columns are the coordinates of the $\beta'$ basis when expressed in the old basis $\beta$. Setting $Q = {}_\gamma I_{\gamma'}$, we see that

$${}_{\gamma'}T_{\beta'} = Q^{-1}({}_\gamma T_\beta)P.$$