The Gaussian Elimination algorithm.

Input: A list of vectors $(v_1, \dots, v_k)$ in V and a basis $(e_1, \dots, e_n)$ of V.

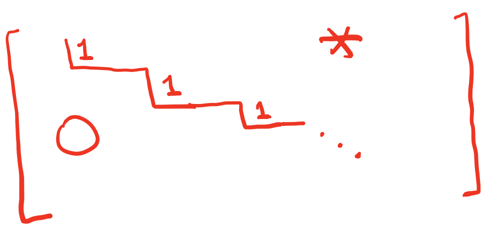
we can write input data as follows: $\begin{cases} v_1 = a_{11} e_1 + \dots + a_{1n} e_n \\ \vdots \\ v_k = a_{k1} e_1 + \dots + a_{kn} e_n \end{cases}$

or more succinctly as a $k \times n$ matrix of numbers $a_{ij} \in \mathbb{F}$, $\begin{matrix} 1 \le i \le k \\ 1 \le j \le n \end{matrix}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & \ddots & \vdots \\ a_{k1} & & \cdots & a_{kn} \end{bmatrix}$$

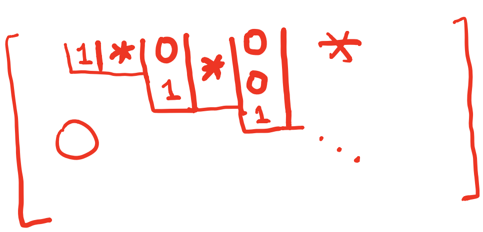
Output: A list of vectors $(w_1, \dots, w_k)$, with the same span as $(v_1, \dots, v_k)$, such that the corresponding matrix is in Row Echelon or Reduced Row Echelon form

RE



first nonzero number in row R_i is 1 in position t_i and $t_{i+1} > t_i$

RRE



same as above, but also, for all i, the t_i position is $= 0$ for all other rows R_j $j \neq i$.

here "$*$" means any numbers in $\mathbb{F}$ and "0" means all zeros. and each "1" entry is called the "echelon" in its row.

The purpose of GE is to make it easy to understand $\mathrm{span}(v_1, \dots, v_k)$.
Can easily answer questions like

- Is $(v_1, \dots, v_k)$ linearly independent?
- What is $\dim \mathrm{span}(v_1, \dots, v_k)$?
- find a basis for $\mathrm{span}(v_1, \dots, v_k)$
- Is $\mathrm{span}(v_1, \dots, v_k) = \mathrm{span}(u_1, \dots, u_\ell)$?
- Solving systems of (in)homogeneous linear equations.

Elementary Row Operations:

GE proceeds by successively applying one of three basic transformations:

1) switching: $(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \xmapsto{R_i \leftrightarrow R_j} (v_1, \dots, v_j, \dots, v_i, \dots, v_k)$.

2) scaling $\lambda \neq 0$ $\quad (v_1, \dots, v_i, \dots, v_k) \xmapsto{R_i \to \lambda R_i} (v_1, \dots, \lambda v_i, \dots, v_k)$

3) shearing $\quad (v_1, \dots, v_i, \dots, v_j, \dots, v_k) \xmapsto{R_i \mapsto R_i + \lambda R_j} (v_1, \dots, v_i + \lambda v_j, \dots, v_j, \dots, v_k)$

Each operation is reversible and does not change the span. Since we are changing the list of vectors, the matrix will change by switching, scaling, or shearing the <u>rows</u>.

Obtaining the RE form: "Forward pass"

Step 1:

- find a v_e with earliest nonzero coefficient a_{ei}, and switch it with v_1. (switch)
- Scale the new v_1 by $(a_{ei})^{-1}$ (scale)
- If any other vector v_m has nonzero a_{mi}, shear it, replacing v_m by $v_m - a_{mi}(a_{ei}^{-1} v_1)$ (shear)

output of Step 1:
$$\begin{cases} \tilde{v}_1 = (a_{ei})^{-1} v_e \\ \tilde{v}_e = v_1 - (a_{1i}\, a_{ei}^{-1}) v_e \\ \tilde{v}_m = v_m - (a_{mi}\, a_{ei}^{-1}) v_e, \quad m \neq 1, e \end{cases}$$

output matrix:
$$\begin{bmatrix} 0 & \begin{matrix} 1 & * & \cdots & * \\ & * & \cdots & * \\ & \vdots & & \vdots \\ & * & \cdots & * \end{matrix} \end{bmatrix}$$

Step 2: Repeat procedure for $(\tilde{v}_2, \ldots, \tilde{v}_k)$

After step 2 matrix looks like

$$\begin{bmatrix} 0 & \begin{matrix} 1 & * & \cdots & \cdots & * \\ & & 1 & * & \cdots & * \\ & & & * & \cdots & * \\ & & & \vdots & & \vdots \\ & & & * & \cdots & * \end{matrix} \end{bmatrix}$$

Step j: Let $(\tilde{v}_1, \dots, \tilde{v}_k)$ be the output of step $j-1$. Repeat the above process with the list $(\tilde{v}_j, \dots, \tilde{v}_k)$

After k steps, the algorithm arrives at the RE form. To get the RRE form, we simply shear off entries above the echelon

RE → RRE: "Backward Pass"

Let $(v_1, \dots, v_k)$ be a list in RE form.

Step 1: Let e be the echelon position of v_k.

use v_k to shear all of $v_1, \dots, v_{k-1}$ so that their coefficient in position e is $= 0$

$$\text{output} \begin{cases} \tilde{v}_1 = v_1 - a_{1e} v_k \\ \vdots \\ \tilde{v}_{k-1} = v_{k-1} - a_{k-1,e} v_k \\ \tilde{v}_k = v_k \end{cases}$$

Step 2: use $\tilde{v}_{k-1}$ to shear $\tilde{v}_1, \dots, \tilde{v}_{k-2}$ in same way, and so on.

after k steps, this terminates in RRE form.