Sets

A set is a collection of objects viewed as an object in its own right.
If an object x is contained in the set S

"x is an element of S"

$$x \in S$$

Ex.:

⓪ $\emptyset$ the empty set.

① $P = \{R, G, B\}$ a set of three letters.

$$= \{G, B, R\}$$

$$= \{R, R, B, G, B\}.$$

② $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$

the set of integers.

③ $\mathbb{Q} = \{p/q : p \in \mathbb{Z},\ q \in \mathbb{Z} \setminus \{0\}\}$

the rational numbers.

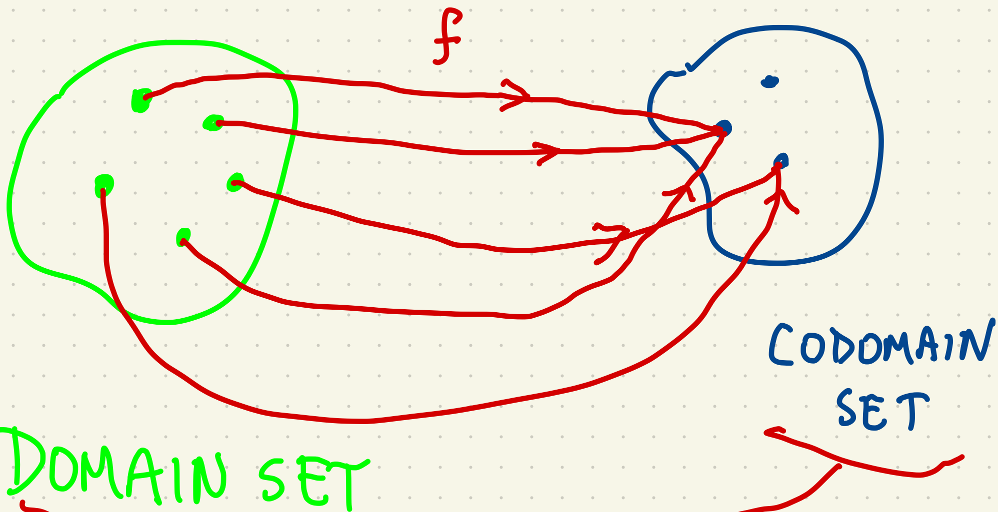
④ $\mathbb{R}$, $\mathbb{C}$ the real or complex numbers

A set is finite when it has a finite number of elements. This number

$$|S| \in \{0, 1, 2, \dots\}$$

is called the "cardinality" of S.

Maps between sets



A map from set X to set Y assigns to each $x \in X$ an element $f(x) \in Y$

$$X \xrightarrow{f} Y$$
$$x \longmapsto f(x)$$

Ex.: (1) A colour in the RGB system is a map

$$P = \{R, G, B\} \longrightarrow \{0, 1, \dots, 255\}$$

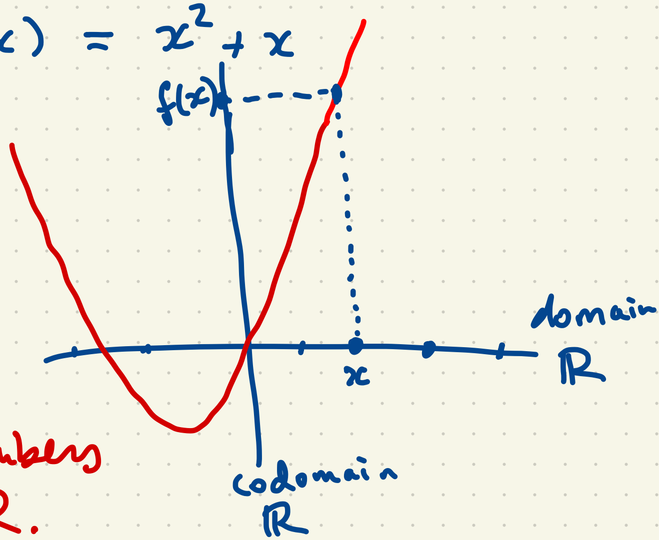
white: $R \longmapsto 255$
$G \longmapsto 255$
$B \longmapsto 255$

red: $R \longmapsto 255$
$G \longmapsto 0$
$B \longmapsto 0$

(2) You studied maps $\mathbb{R} \xrightarrow{f} \mathbb{R}$

such as $f(x) = x^2 + x$

graphing:

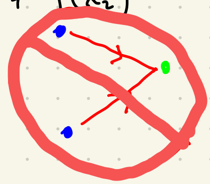


"functions"

codomain = numbers
$\mathbb{R}$.

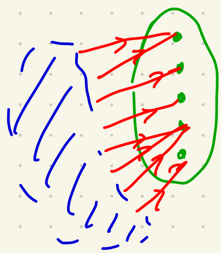
The map $f: X \to Y$ is

- <u>injective</u>: when different inputs result in different outputs
  (f injection) i.e. if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

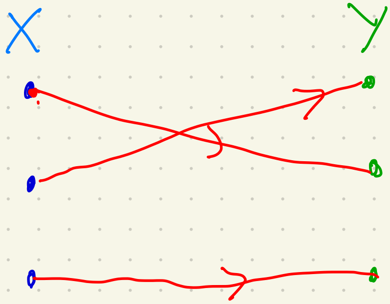


- <u>surjective</u>: when all possible outputs are achieved
  (surjection) i.e. $\forall\, y \in Y \;\; \exists\, x \in X$ s.t. $f(x) = y$

  for all   there exists



- <u>Bijective</u>: both injective and surjective.
  (f bijection)

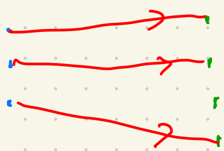


bijection.

surjection

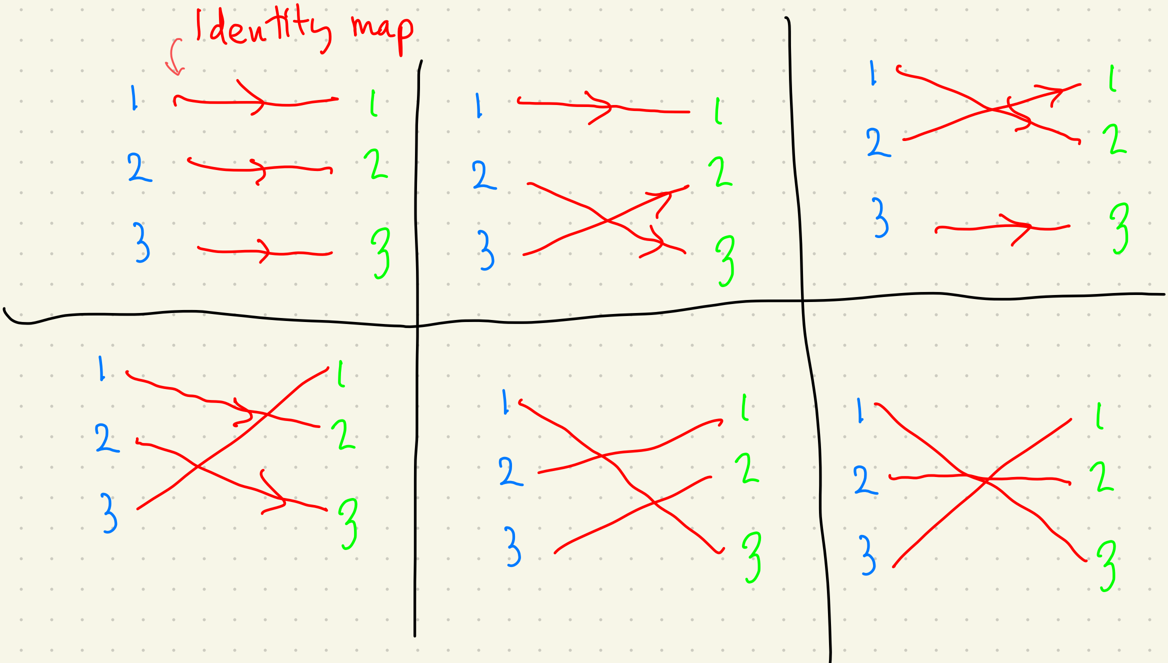
not injective.

Injection.



not surjective.

Ex.: Bijections $\{1, 2, 3\} \longrightarrow \{1, 2, 3\}$.



6 possible bijections $\{1,2,3\} \longrightarrow \{1,2,3\}$

Ex.: When Domain = Codomain of f i.e.

$$f : X \longrightarrow X$$

there is a specific map called the identity map $\mathrm{Id}_X = I_X$

$$\mathrm{Id}_X : x \longmapsto x \qquad \forall x \in X$$

Composition of Maps (Definition)

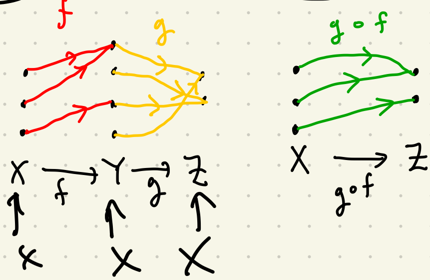
The composition of $f: X \to Y$ and $g: Y \to Z$ (defined only if $\text{codom}(f) = \text{dom}(g)$) is a map

$(g \circ f): X \longrightarrow Z$ which takes $x \in X$ to $g(f(x)) \in Z$.

$f \circ g$ undefined

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Ex.:



$$\begin{tikzcd} X \arrow[loop left, "g\circ f"] \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \arrow[loop right, "f\circ g"] \end{tikzcd}$$

$g \circ f : X \xrightarrow{f} Y \xrightarrow{g} X$

$f \circ g : Y \xrightarrow{g} X \xrightarrow{f} Y$

Q.: Prove that composition is Associative

i.e. for maps $S \xrightarrow{f} T \xrightarrow{g} U \xrightarrow{h} V$

$$h \circ (g \circ f) \underset{\text{same map}}{=} (h \circ g) \circ f$$

Assoc. Law

At this point, we have Sets, Maps between sets, and composition of Maps! Defines Category of Sets !

Deliberate confusion

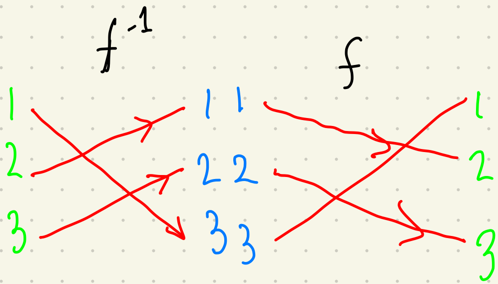
purists: draw in reverse.

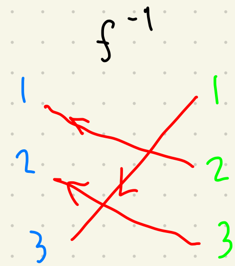
$$\begin{tikzcd} Z & Y \arrow[l, "g"'] & X \arrow[l, "f"'] \arrow[ll, bend left, "g \circ f"] \end{tikzcd}$$

Inverses of maps: the inverse of a map $f: X \to Y$ is, by defn: a map $g: Y \to X$ s.t.

$$\begin{cases} g \circ f = I_X \\ f \circ g = I_Y \end{cases} \qquad \left(\neq \text{"}\frac{1}{f}\text{"} \right)$$

a map f may not have an inverse but if it does, we call it f^{-1}





has inverse

The bijections from a set X to itself $\mathrm{Bij}(X,X)$ is a very special set of maps:

- any two of them may be composed to give a third, gives an associative multiplication on $\mathrm{Bij}(X,X)$

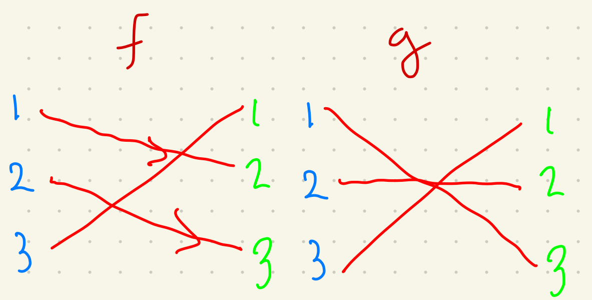
- $Bij(X,X)$ contains a special element I_X which acts as multiplicative identity
- every element in $Bij(X,X)$ has an inverse,

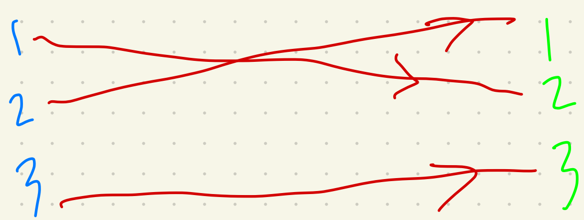
$$\Rightarrow \left(Bij(X,X), \overset{\text{composition}}{\circ}, I_X\right)$$ is a <u>group</u>

called S_X the permutations or symmetries of the set X.

<u>Binary operation on set X</u>

$$X \times X \xrightarrow{m} X \qquad \times$$
$$(a,b) \longmapsto m(a,b) = a \cdot b \qquad +$$





$g \circ f$

Subsets

Def$^{\underline{n}}$: Given a set Y, a <u>subset</u> $X \subseteq Y$ is a set comprising some (possibly none, or all) of the elements of Y.

can think of X as the elements of Y satisfying some constraint.

<u>Ex:</u> $\mathbb{Z}$ set of integers

such that

$$X \subseteq \mathbb{Z} \qquad X = \{ n \in \mathbb{Z} : n \text{ is odd and } 1 \leq n \leq 6 \}$$
$$= \{1, 3, 5\}.$$

If $Y \xrightarrow{f} Z$ and $X \subseteq Y$ then the <u>restriction</u> of f to X is a map

$$\begin{aligned} X &\xrightarrow{f|_X} Z \\ x &\longmapsto f(x) \end{aligned}$$

<u>About Subsets:</u> The set of all subsets of X is called $\mathcal{P}(X)$, the Power set of X

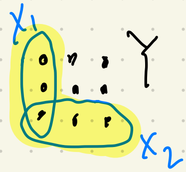
$$\mathcal{P}(\{R, G, B\}) = \{\phi, \{R,G,B\}, \{R\}, \{G\}, \{B\}, \{R,G\}, \{R,B\}, \{G,B\}\}$$

set of cardinality 8

Basic operations on subsets $X \subseteq Y$.

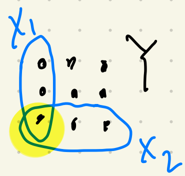
- $X_1 \cup X_2 = \{x \in Y : x \in X_1 \text{ or } x \in X_2\}$

  union of subsets



- $X_1 \cap X_2 = \{x \in Y : x \in X_1 \text{ and } x \in X_2\}$

  intersection of subsets



(Both of these are "Binary operations" on $\mathcal{P}(Y)$.)

Subsets coming from a map $f : X \to Y$

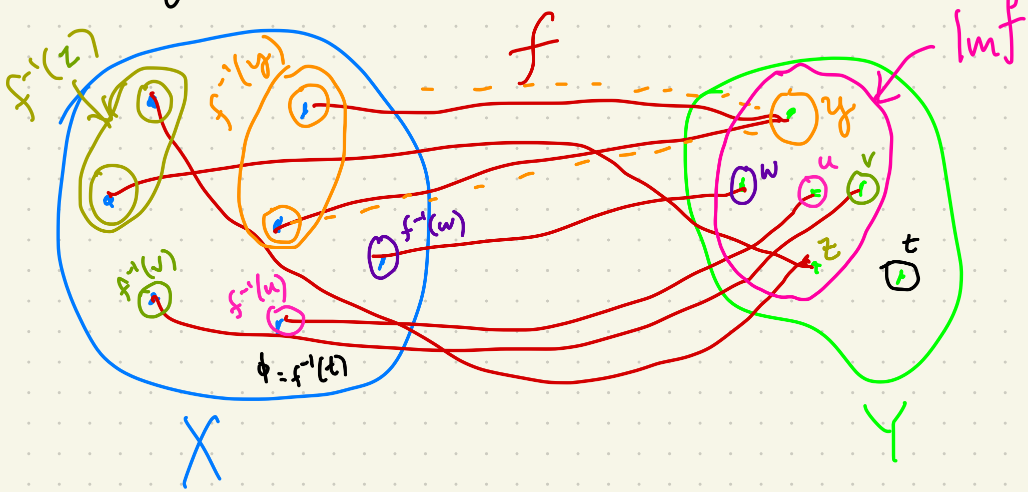
Def^n: The Image of f, $\mathrm{Im}(f)$, is the subset of Y defined by

$$\mathrm{Im} f = \{ y \in Y : \exists\, x \in X \text{ with } f(x) = y \}$$

<u>Def^n</u>: If $f: X \longrightarrow Y$ and we fix an element $y \in Y$ then its <u>PREIMAGE</u> is

not 'inverse map'

$$f^{-1}(y) = \{x \in X \text{ s.t. } f(x) = y\}$$



This means that for each $y \in Y$ we get a subset $f^{-1}(y) \subseteq X$.

==this defines a Partition of domain into subsets labeled by the elements of Im f.==

given a map $f: X \to Y$, we obtain:

① $\mathrm{Im} f = \{ y \in Y : \exists\, x \in X \text{ with } y = f(x) \}. \subseteq Y$

② Partition of X into preimages:

$$P = \{ f^{-1}(y) : y \in \mathrm{Im} f \}$$

Partition of X labeled by $\mathrm{Im} f$.

<u>Note:</u> ① there is a natural map from X to P:

$$X \xrightarrow{\pi} P$$
$$x \longmapsto f^{-1}(f(x))$$

<u>surjective!</u>

② There is a natural map. $P \longrightarrow \mathrm{Im} f$

Bijection

$$f^{-1}(y) \longmapsto y$$

②' There is a natural map $P \xrightarrow{j} Y$

$$f^{-1}(y) \longmapsto y$$

<u>Injective!</u>

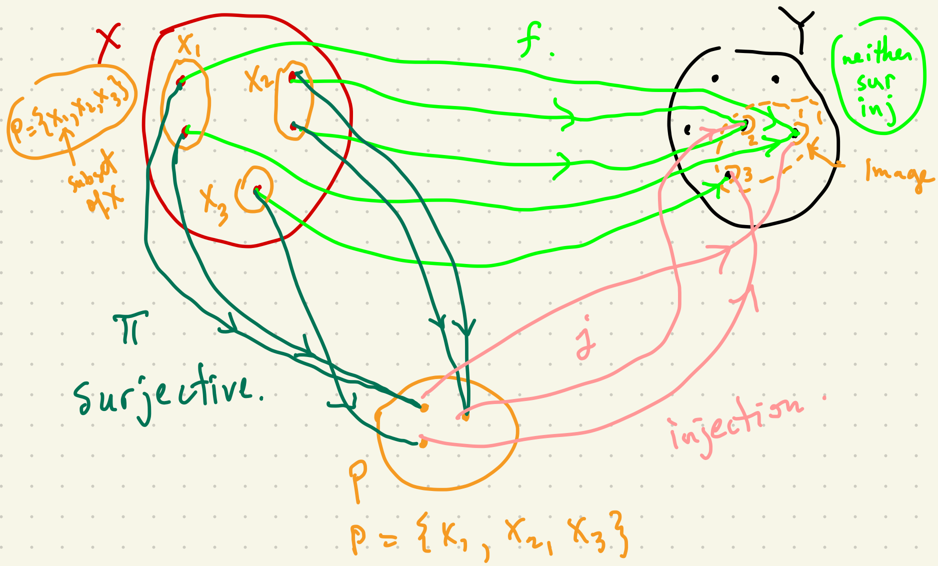
③ If we compose, we get $x \mapsto f(x)$!

$$\begin{tikzcd} X \arrow[r, two heads, "\pi"'] \arrow[rr, bend right, "f"'] & P \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

$f = j \circ \pi$

Factorization of f into a surjection followed by injection.

Prop: Any map $f: X \to Y$ may be factorized $f = j \circ \pi$, π surjective and j injective.



$$\left(A \text{ if } \text{ and } \text{ only if } B \right)$$

if: $B \Rightarrow A$

only if: $A \Rightarrow B$

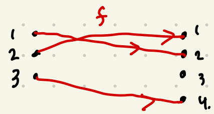
$$A \iff B$$

Explicit description of maps

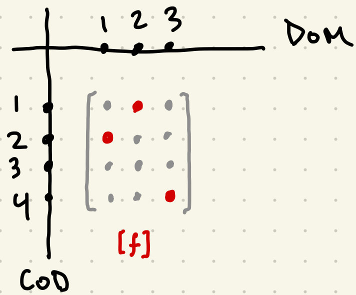
The standard set of n elements $n = 0, 1, 2, \ldots$

$$B_n = \{1, 2, 3, \ldots, n\} \qquad B_0 = \phi.$$

Instead of writing a map $B_m \xrightarrow[f]{} B_n$ as follows



we can encode f as follows : as a "graph".



$$[M_f] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4×3 grid of numbers

matrix rep. of map

$f : B_3 \longrightarrow B_4$

the representation of f as a 4×3 matrix uses the fact that B_n has a preferred order.
(we put the domain and codomain in specific order on graph)

<u>Note:</u> permute codomain $\Longrightarrow$ permute rows

permute domain $\Longrightarrow$ permute columns

Cartesian Product

given sets X, Y, their (Cartesian) product $X \times Y$ is defined as follows

$$X \times Y = \{ \underbrace{(x,y)}_{\text{ordered pair.}} : x \in X \text{ and } y \in Y \}.$$

(list of length 2)

$(1,2) \neq (2,1)$
$\{1,2\} = \{2,1\}$

eg. $P = \{R, G, B\}$

$$B_2 \times P = \{ (1,R), (2,R), (1,G), (2,G), (1,B), (2,B) \}$$

similarly if $X_1, \cdots, X_k$ are sets

$$\underbrace{X_1 \times X_2 \times \cdots \times X_k}_{\prod_{i=1}^{k} X_i} = \{ \underbrace{(x_1, \cdots, x_k)}_{k\text{-tuple or list of length } k} : x_i \in X_i \ \forall i \}$$

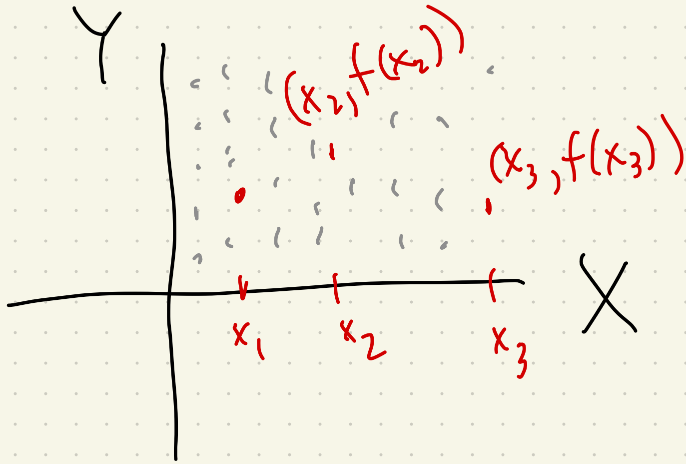
Special case $\overbrace{X \times \cdots \times X}^{k} = X^k$

Def: The graph of a map $f : X \to Y$ is the subset

$$\Gamma_f = \{ (x, y) \in X \times Y : y = f(x) \}$$

$\parallel$

Graph(f)



$f : X_1 \times X_2 \to Y$.

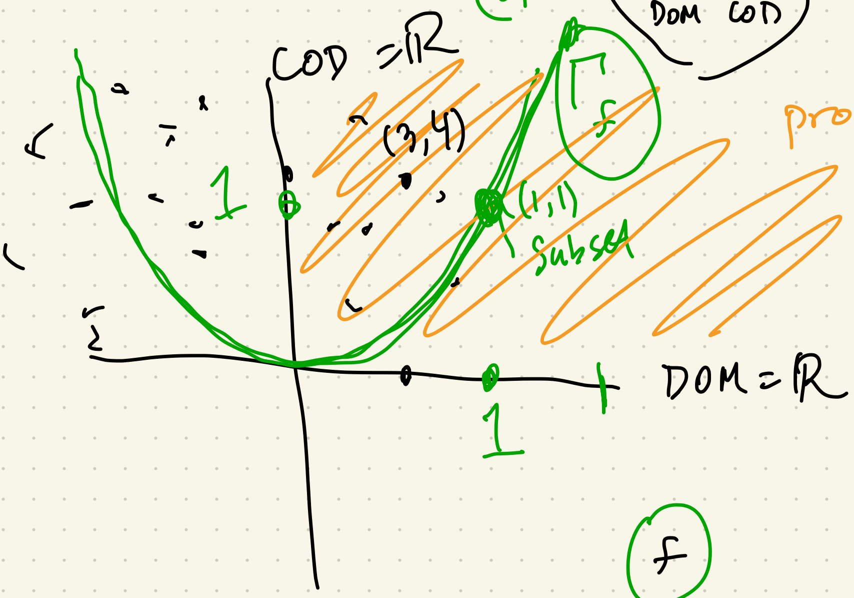
$$
\begin{array}{ccccc}
B_1 & & B_2 & & B_2 \\
\{1\} & \hookrightarrow & \{1,2\} & \longrightarrow & \{1,2\} \\
1 & \longmapsto & 1. & & \\
 & & 1 & \longmapsto & 1 \\
 & & 2 & \longmapsto & 2.
\end{array}
$$

surjective

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$\underset{\text{DOM}}{\mathbb{R}} \times \underset{\text{COD}}{\mathbb{R}} = \mathbb{R}^2$ notation

product.



$$
f : \mathbb{R} \longrightarrow \mathbb{R}
$$

$$
x \longmapsto x^2
$$

Technically, a map $f: X \to Y$ may be defined in terms of its graph:

Alternate definition: a map $f: X \to Y$ is a subset Γ_f of $X \times Y$ satisfying the "vertical line test", i.e.

$$\forall\, x \in X,\ \exists!\ y \in Y \text{ such that } (x,y) \in \Gamma_f$$

for all ... there exists a unique ...

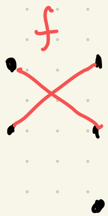
Q: Does there exist a map from the empty set to itself?

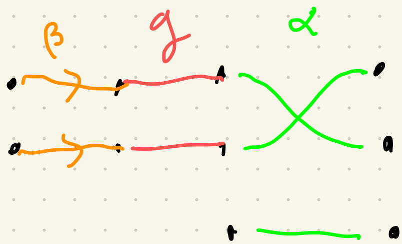
Def: Let f, g be maps $B_m \to B_n$

$f \sim g$ "f is similar to g" when

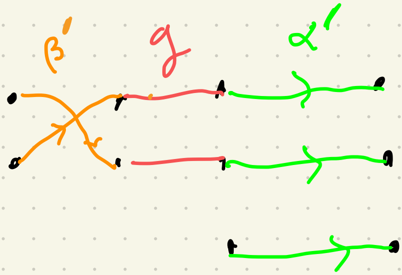
we can find relabelings α, β of Cod, Dom.

s.t. $\alpha \circ g \circ \beta = f$





$\alpha \circ g \circ \beta = f$



$\alpha' \circ g \circ \beta' = f$

Note: $\sim$ is an example of an

EQUIVALENCE RELATION on a set S

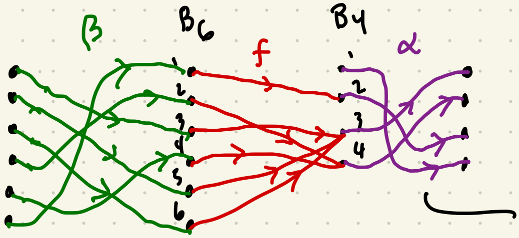
i) $x \sim x \quad \forall x \in S$ Reflexive

ii) $x \sim y \Leftrightarrow y \sim x \quad \forall (x,y) \in S \times S$ Symmetric

iii) if $x \sim y$ and $y \sim z$ then $x \sim z \quad \forall (x,y,z) \in S^3$ transitive

(EQUIVALENCE relations $\leftrightarrow$ partitions !)

Q.: prove that similarity of maps is an equivalence relation



$\alpha \to$ row permutat.

$\beta \to$ column permutation

codomain relabeling

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$M(\alpha \circ f \circ \beta)$

$$\begin{array}{c|cccccc} \text{COD} \downarrow \backslash \text{DOM} \to & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 1 & 1 \\ 4 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}$$

4×6 matrix

$\parallel$

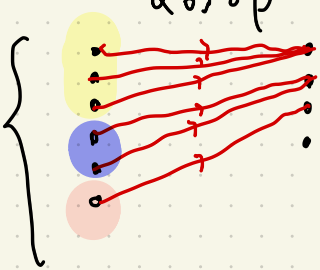
$M(f)$

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

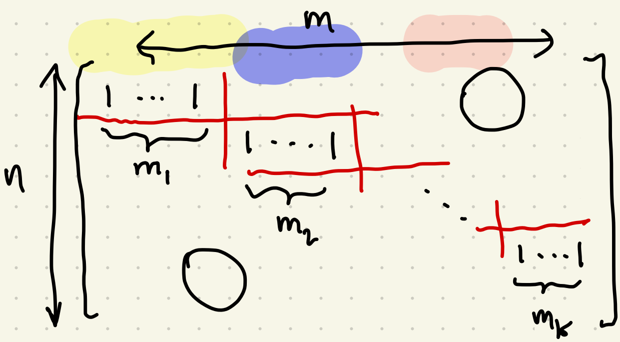
$M(\alpha \circ f)$

$\alpha \circ f \circ \beta$

partition of domain



Thm: by relabeling dom & cod, any map $B_m \to B_n$ is similar to one in standard form:



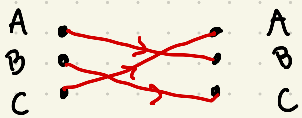
$m_1 \geq m_2 \geq \cdots \geq m_k$

$m_1 + \cdots + m_k = m$

is a partition of domain.

Special case: Bijections from X to itself e.g. $B_n \to B_n$

Labeling a self-map: once we choose a labeling of the domain



i.e. $\left(\begin{array}{l} \alpha : \{A,B,C\} \to B_3 \\ \text{bijection} \end{array} \right)$

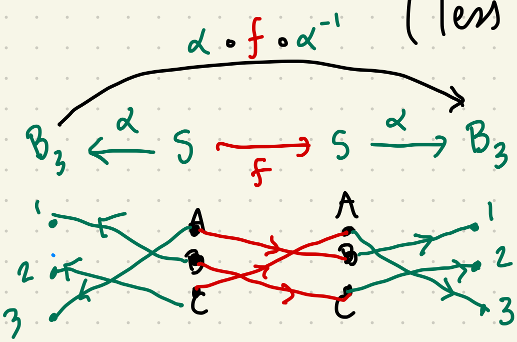
"labeling"

this automatically gives a label for codomain!!

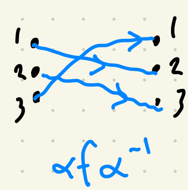
i.e. we do not have two indep. choices of labeling!

(less freedom).

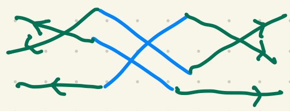
$\{A, B, C\} = S$

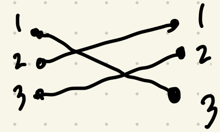


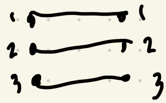
So, by choosing labeling $\alpha : S \to B_3$, we obtain the map $\alpha \circ f \circ \alpha^{-1} : B_3 \to B_3$



If we relabel, this takes effect on both dom & cod.





cannot relabel to obtain  unless we allow independent relabeling of DOM + CODOMAIN.

two equivalence relations

(1) for maps $f : X \to Y$

say f, f' are similar when

Bijection.

$\exists \quad \alpha : Y \xrightarrow{\cong} Y$ and $\beta : X \xrightarrow{\cong} X$

with $f' = \alpha \circ f \circ \beta$.

(2) for maps $f : X \longrightarrow X$

say f, f' are conjugate

if $\exists \quad \alpha : X \xrightarrow{\cong} X$ s.t.

$$f' = \alpha \circ f \circ \alpha^{-1}$$

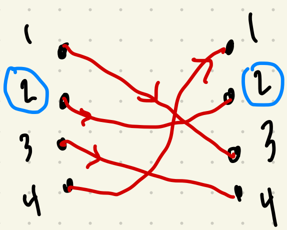
Classify Bijections $B_n \to B_n$ up to conjugation

① A fixed point of $f: X \to X$ is $x \in X$ with $f(x) = x$

② A <u>cycle</u> of $f: X \to X$ is a subset $S \subseteq X$ of the form

$$S = \{x, f(x), f(f(x)), \dots\}$$

Ex:



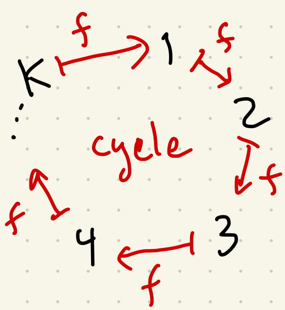
2 is a fixed pt $f(2) = 2$

$\{1, 3 = f(1), 4 = f(3)\}$

cycle of length 3.

in a cycle, can label

$$x \longmapsto 1$$
$$f(x) \longmapsto 2$$
$$\vdots$$
$$f^k(x) \longmapsto k+1$$



If $f : B_k \to B_k$ has a cycle labeled as above

$f = (123)(456)(78)$

$$
\begin{array}{c c}
 & \xleftarrow{\quad k \quad}\!\!\xrightarrow{\quad} \\
M(f) = \begin{array}{c}\uparrow\\ k \\ \downarrow\end{array} &
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & & & 0 \\
0 & 1 & 0 & & & \vdots \\
0 & 0 & 1 & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
0 & 0 & \cdots & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

$M(f)$

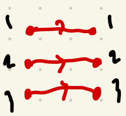
$$
\begin{array}{c|ccc|ccc|cc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 0 & 0 & 1 & & & & & \\
2 & 1 & 0 & 0 & & 0 & & & 0 \\
3 & 0 & 1 & 0 & & & & 0 & 0 \\
\hline
4 & 0 & 0 & 0 & 0 & 0 & 1 & & \\
5 & & 0 & 0 & 1 & 0 & 0 & & 0 \\
6 & & & & 0 & 1 & 0 & & \\
\hline
7 & & 0 & & & 0 & & 0 & 1 \\
8 & & & & & & & 1 & 0
\end{array}
$$

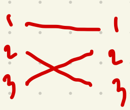
Classification: Any bijective self-map decomposes into cyclic parts. (cycle of length 1 = fixed pt).

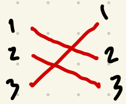
This is why elements of the group

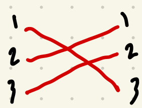
$$S_n = \mathrm{Bij}(B_n, B_n) \text{ are usually written}$$

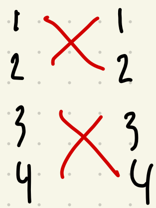
in "cycle notation"


↔ "1"


↔ "(2 3)"


↔ "(1 2 3)"


↔ "(1 3 2)"


(12)(34)

(32)(14)

(··)(··)


(1234)