Vector Space

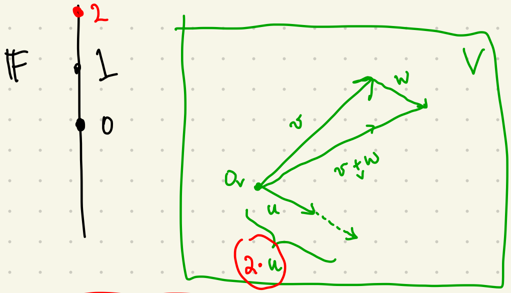
1^{st} component of v.space is the number system used: a field $\mathbb{F}$

Def^n

A vector space over the $\underbrace{\text{field } \mathbb{F}}_{\text{scalars}}$ $(\mathbb{F}, +_F, 0_F, \cdot_F, 1_F, inv_F \ldots)$ is a set V

equipped with abelian group structure $(V, +_V, 0_V, i_{+V} \ldots)$

$+_V$: vector addition. 0_V: origin



$v \in V$

introduce new kind of operation "scaling a vector by a scalar/number."

$$\boxed{\begin{array}{l} \text{Additional structure} \\ \cdot_s \quad \text{binary operation} \quad \mathbb{F} \times V \xrightarrow{\cdot_s} V \\ \qquad\qquad\qquad\qquad\quad (a, u) \longmapsto a \cdot_s u \end{array}}$$

axioms satisfied by $\cdot_s$ are:

1. compatibility $\cdot_s$ $\cdot_{\mathbb{F}}$: $\quad (a \cdot_F b) \cdot_s u = a \cdot_s (b \cdot_s u)$
2. compatibility $\cdot_s$ $+_V$: $\quad a \cdot_s (u +_V v) = a \cdot_s u +_V a \cdot_s v$
3. relation $\cdot_s$, $+_{\mathbb{F}}$, $+_V$: $\quad (a +_F b) \cdot_s u = a \cdot_s u +_V b \cdot_s u$
4. $1 \in \mathbb{F}$ acts on V via identity map : $\quad 1_F \cdot_s u = u$

Consequence of ③ $a, b = 0$

$$\underset{\displaystyle 0 \cdot u}{\underset{\parallel}{(0+0) \cdot u}} = 0 \cdot u + 0 \cdot u$$

use additive inverse in V

$\Downarrow$

$$\boxed{0_V = \overline{0_F \cdot_s u} \quad \forall u}$$

Examples of Vector spaces

fix $\mathbb{F}$ field

① $\underbrace{V = \{0\}}_{\text{trivial group.}}$

$$\overset{\mathbb{F}}{\overset{\downarrow}{a}} \cdot 0 = 0 \quad \forall a.$$

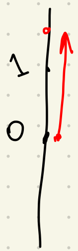
"the zero / trivial vector space over $\mathbb{F}$"

② $\mathbb{F}$ itself!

$$\left(\begin{array}{l} V = \mathbb{F} \\ 0_V = 0_{\mathbb{F}} \\ + = +_{\mathbb{F}} \end{array} \right)$$

$$\underset{\mathbb{F}}{\underset{\uparrow}{a}} \cdot_s \underset{V=\mathbb{F}}{\underset{\uparrow}{b}} = a \cdot_F b$$

use pre-existing multiplic.



" A 1-dimensional vector space over $\mathbb{F}$ "

③ $V = \mathbb{F} \times \mathbb{F}$

"2-dimensional vector space over $\mathbb{F}$"

$0_V = (0_F, 0_F)$

$$(a,b) +_V (c,d) = (a +_F c, b +_F d)$$

"component-wise addition"

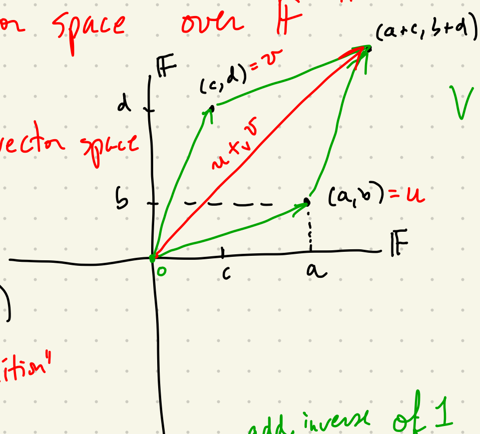
add. inverse

$$(a,b) +_V (\underset{\uparrow}{-a}, -b) = (0,0),$$

$a \in \mathbb{F}$

$$\underset{\text{inv}_{+_F}(a)}{\overset{\parallel}{-a}} = (\underset{\text{mult. identity}}{\underset{\uparrow}{-1}}) \cdot_F a$$

add. inverse of 1



need scalar mult.

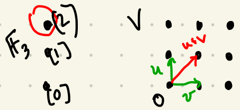
$$\mathbb{F} \times \overset{V}{(\mathbb{F} \times \mathbb{F})} \xrightarrow{\cdot_s} \overset{V}{(\mathbb{F} \times \mathbb{F})}$$

$$a \cdot_S (b,c) = (a \cdot_F b, \; a \cdot_F c)$$

"componentwise multiplication"

(3) $\mathbb{F}_3 \times \mathbb{F}_3 = V$ vector space over $\mathbb{F}_3$.

$[3] = [0]$



discrete 3×3 plane.

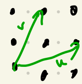
$u = ([0],[1])$

$v = ([1], 0)$

$u + v = (1,1)$

$\mathbb{Z}_3$

integers mod 3



$u = ([2],[1])$

$v = ([1],[2])$

$u + v = ([3],[3])$
$= (0,0)$.

$2 \cdot u = 2 \cdot (2,1)$
$= (4,2) = (1,2)$

in this vector space $(1,2) + (2,1) = (0,0)$

additive inverse of

$(2,1) = -(1,2)$

$(1,2) = 2 \cdot (2,1)$.

(4) similarly in higher "dimension"

$\mathbb{F}$ $\qquad V = \mathbb{F}^n = \underbrace{\mathbb{F} \times \mathbb{F} \times \mathbb{F} \cdots \times \mathbb{F}}_{n.}$ $\quad n = 1, 2, 3, \ldots$

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$$

$$\lambda \cdot_s (a_1, \ldots, a_n) = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n)$$

$\lambda \in \mathbb{F}$

(5) $\mathbb{F}$ any field $\qquad V = P(\mathbb{F}) = \mathbb{F}[x]$

the vector space of polynomials with coeffs in $\mathbb{F}$

$$\mathbb{F}[x] = \left\{ a_0 + a_1 x + \cdots + a_n x^n \;:\; \begin{array}{l} n \in \{0, 1, \ldots,\} \\ a_i \in \mathbb{F} \end{array} \right\}.$$

vector addition is termwise (addition of polynomials)
scalar mult is termwise.

"infinite - dimensional" (not using $\cdot$ on polynomials at all).

Warning: For some fields, it is not valid to view polynomials as functions $\mathbb{F} \xrightarrow{p} \mathbb{F}$.

A polynomial $\mathbb{F}[x] \ni p = a_0 + a_1 x + \cdots + a_n x^n$

defines a map $(x \in \mathbb{F}) \longmapsto a_0 + a_1 \cdot x + \cdots + a_n x^n$,

but for $\mathbb{F} = \mathbb{F}_2 = \mathbb{Z}_2 = \{0,1\}$, $p_1 = 0$, $p_2 = x^2 + x$

are different polynomials but define same function.

as a function $\{0,1\} \longrightarrow \{0,1\}$

$0 \longmapsto 0$ $\quad 1^2 + 1 = 1+1 = 0$ in $\mathbb{F}_2$.

$1 \longmapsto 0$

The map from polynomials to functions $\mathbb{F} \to \mathbb{F}$ is not always injective!

6) $V = \mathbb{F}^\infty = \{ (x_1, x_2, \cdots) : x_i \in \mathbb{F} \}$ "sequences" "infinite list"

$+_v$ is component wise $\quad \cdot_s$ is

$\lambda \cdot_s (x_1, x_2, \cdots) = (\lambda x_1, \lambda x_2, \cdots)$

7) function spaces.

Let X be a set, $\mathbb{F}$ a field

The set of all maps $X \xrightarrow{f} \mathbb{F}$ $= \mathbb{F}^X$

$$V = \mathbb{F}^X = \{ f : X \longrightarrow \mathbb{F} \text{ a map} \}$$

has the structure of a v.sp. over $\mathbb{F}$

$0 \in \mathbb{F}^X$ is the f^n taking value 0 $\forall x \in X$

$f_1 + f_2$ is the f^n $x \longmapsto f_1(x) \underset{\mathbb{F}}{+} f_2(x)$

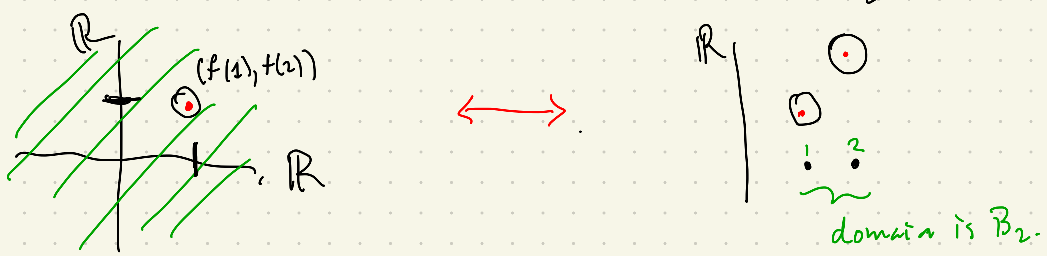
$\underset{\mathbb{F}}{\underset{\cap}{\lambda}} \cdot \underset{\mathbb{F}^X}{\underset{\cap}{f}}$ = the f^n $x \longmapsto \lambda \underset{\mathbb{F}}{\cdot} f(x)$.

eg.: $B_n = \{1, 2, \dots, n\}$ $\qquad \mathbb{F}^{B_n} = \{ f : B_n \longrightarrow \mathbb{F} \}$

f can be written as a list of it values on $1, 2, \dots, n$.

$(f(1), f(2), \dots, f(n)) \in \mathbb{F}^n$ $\qquad$ ($\mathbb{F}^{B_n}$ is same v.sp. as $\mathbb{F}^n$)

$\mathbb{F} \times \mathbb{F}$ is an example of a function space:
can view it as the f^{ns} on the set $B_2 = \{1, 2\}$



Subspaces

Let V be a vector space over $\mathbb{F}$.

A linear subspace $U \subseteq V$ is a subset which inherits v.space str. from V.

i.e.: 1. $0_V \in U$

2. $u_1, u_2 \in U \Rightarrow u_1 + u_2 \in U$. U closed under vector addition

3. $\lambda \in F$ $u \in U \Rightarrow \lambda \cdot u \in U$. U closed under scalar mult.

do we have to check that additive inverses $\in U$?

no, since add. inverse of any $v \in V$ is $(-1)\cdot_s v$

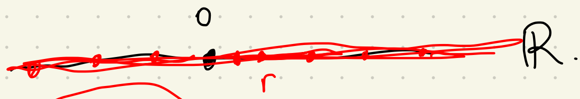
$\left(v + (-1)\cdot_s v = 0_V \right.$ follows from axioms (check!) $\left.\right)$

axioms hold since we assume they do in V already.

Lemma: $0 \in \mathbb{F}$ $v \in V$

$$\boxed{0 \cdot_s v = 0_V}$$

Examples 1)



V.sp. over $\mathbb{R}$.

$\{0\} \subseteq \mathbb{R}$ is a subspace.

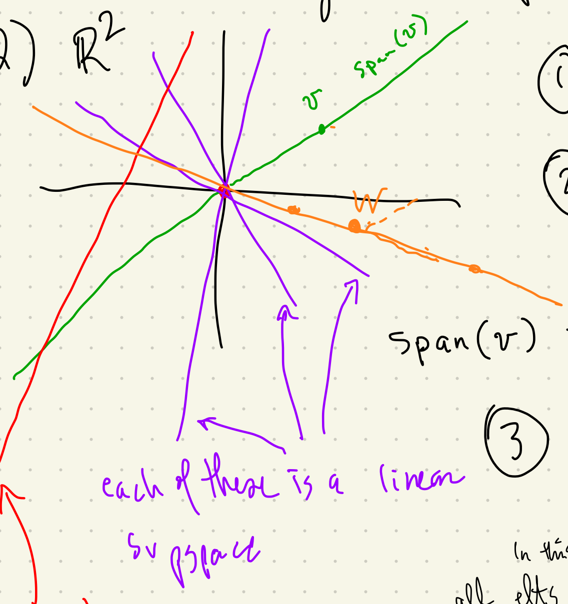
if we include another element $r \in \mathbb{R} \setminus \{0\}$ into U i.e. If U contains 0, $r \neq 0$,

- it must contain $r+r$, $r+r+r$, ...
- " " " $\lambda \cdot r$ $\forall \lambda \in \mathbb{R}$.

$\Rightarrow$ subspace must contain all $\mathbb{R}$. $\Rightarrow$ $U = \mathbb{R}$.

there are only 2 subspaces of v.sp. $\mathbb{R}$: $\{0\}$, $\mathbb{R}$.

2) $\mathbb{R}^2$



each of these is a linear subspace

does not include 0, not closed under +,

① $\{0\}$

② If U contains $v \neq 0$ then it must contain

$$\text{span}(v) = \{\lambda \cdot v : \lambda \in \mathbb{R}\} = \text{straight line through } (0,0)$$

③ Suppose U contains $v \neq 0$ and one other vector $w \notin \text{span}(v)$.

In this case, U must contain all elts of the form $\lambda_1 v + \lambda_2 w$ $\lambda_1, \lambda_2 \in \mathbb{R}$. $\Rightarrow$ all pts in $\mathbb{R}^2$ lie in U.

$$\text{Span}(v, w) = \{\lambda_1 v + \lambda_2 w : \lambda_1, \lambda_2 \in \mathbb{F}\}.$$