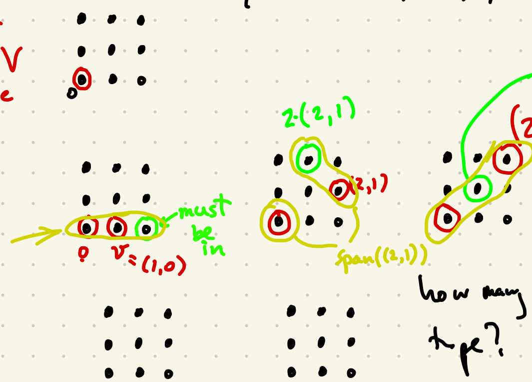
Subspaces & Linear independence

$\mathbb{F}_3^2$ v.space over field $\mathbb{F}_3 = \{0,1,2\}$

$\mathbb{F}_3^2 = \{(a,b) \in \mathbb{F}_3 \times \mathbb{F}_3\}$.

minimal $\{0\} \subseteq V$ subspace



$\frac{1}{2}\cdot(2,2) = 2\cdot(2,2)$
$= (4,4) = (1,1)$

$\mathrm{Span}((2,2)) = \mathrm{Span}((1,1))$

how many subspaces are there of this type?

① choose a nonzero vector v in $\mathbb{F}_p^2$ there are p^2-1 of these

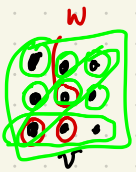
② take scalar multiples of it
⇒ besides zero there are $p-1$ distinct multiples of v.

the # of these subspaces is

$$\frac{p^2-1}{p-1} \qquad \frac{\#\text{ of nonzero vectors}}{\#\text{ of nonzero vectors in } \mathrm{span}(v),\ v\neq 0}$$

$= p+1$

$= 3+1 = 4$



choose $w \notin \mathrm{Span}(v)$.

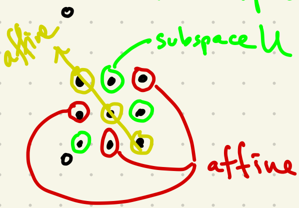
$\mathrm{Span}(v\neq 0,\ w \notin \mathrm{span}(v)) = \mathbb{F}_3^2$

check.

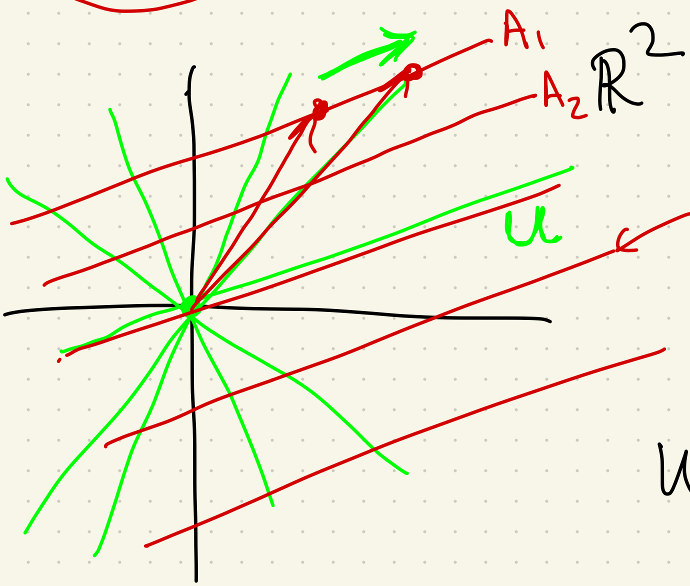
$1+1+4 = 6$

affine subspace ← does not inherit v.space str.

subspace. ← inherits str. of a v.space



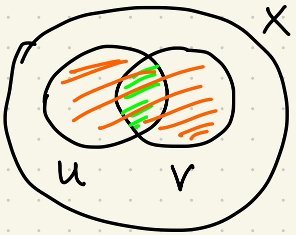
affine subspace modeled on U.



affine lines modeled on U.

$U \subseteq V$
subspace

Operations on subspaces:



U, V subsets of X

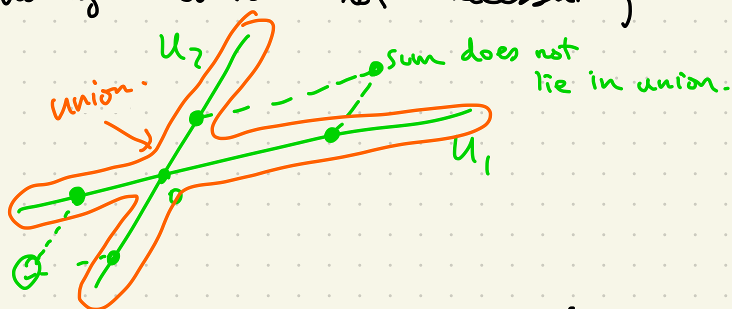
generate new subsets:

- intersection $U \cap V$ "and"
- union $U \cup V$ "or"

Analog for vector subspaces: if $U_1, U_2 \subseteq V$ are subspaces

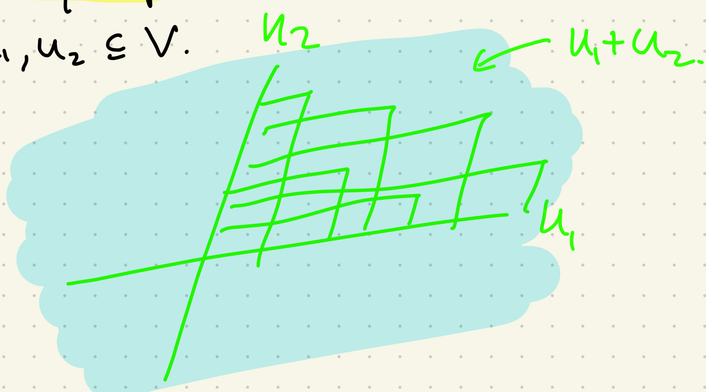
have ① intersection $U_1 \cap U_2 \subseteq V$ is a subspace

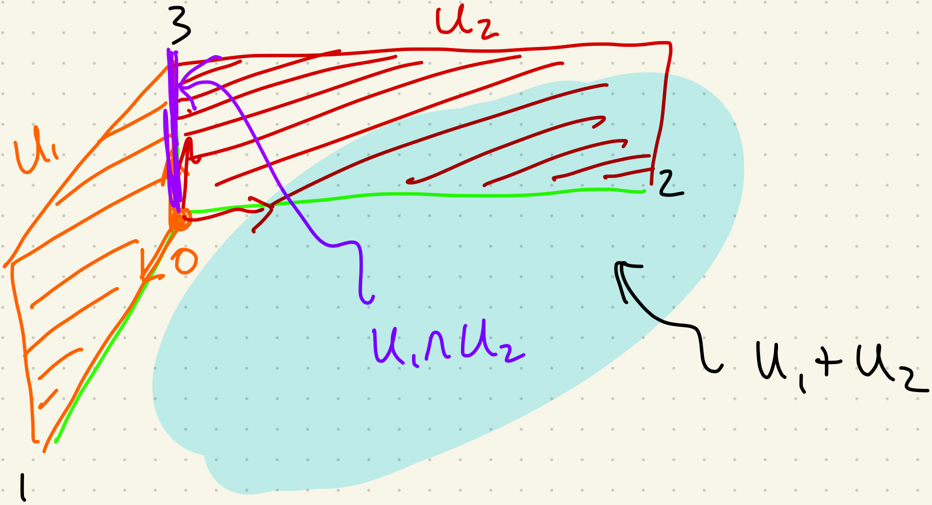
② Warning: union not necessarily a subspace



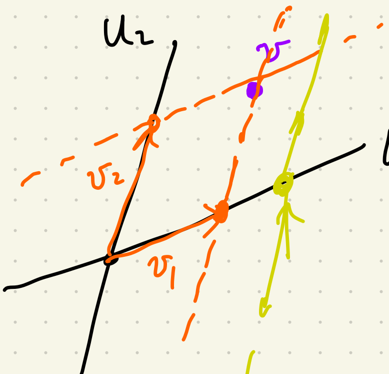
②' Sum of subspaces $U_1 + U_2 = \{ u_1 + u_2 : u_1 \in U_1, u_2 \in U_2 \}$

$U_1, U_2 \subseteq V$.





Property that a sum $U_1 + U_2$ may or may not have: being <u>DIRECT</u>.



$\underbrace{v_1 + v_2 = v.}_{}$

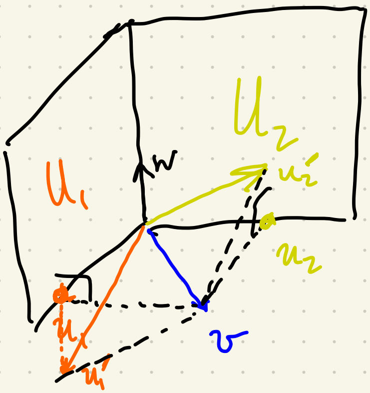
linear comb, $v_1 \in U_1$
$v_2 \in U_2$.

Def^n

$U_1 + U_2$ is DIRECT (denoted by $U_1 \oplus U_2$)

when any $v \in U_1 + U_2$ has <u>unique</u> expression

$$v = u_1 + u_2 \qquad u_1 \in U_1, \quad u_2 \in U_2.$$



$$\boxed{\begin{aligned} u_1 + u_2 &= v \\ u_1' + u_2' &= v \end{aligned}}$$

decomp. of v NOT unique so $U_1 + U_2$ NOT DIRECT

where $u_1' = u_1 + w$
$u_2' = u_2 - w$
$w \in U_1 \cap U_2$

$U_1 + U_2 = \mathbb{R}^3$

Span takes list $(v_1, v_2, \ldots, v_n)$ of vectors in V

outputs a linear subspace $\text{Span}(v_1, \ldots, v_n)$ of V.

Defn

$$\text{Span}(v_1, \ldots, v_n) = \{ a_1 v_1 + \cdots + a_n v_n : a_i \in \mathbb{F} \}$$

$$\text{Span}(\,) = \{0\}$$

empty list

Note: $\text{Span}(v_1, \ldots, v_n) = \text{Span}(v_1) + \text{Span}(v_2) + \cdots + \text{Span}(v_n)$

"list is redundant"

Def^n A list $(v_1, \dots, v_n)$ is <u>linearly dependent</u>

when there exists $a_1, \dots, a_n$, not all zero,

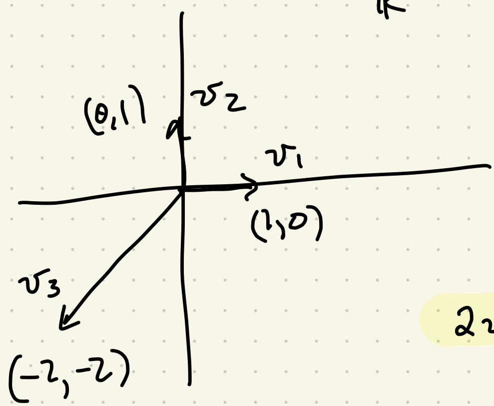
with $$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$

("a nontrivial linear combination equals zero")

If a list is not linearly dep. we say it is

Linearly <u>INDEPENDENT</u>.

$\mathbb{R}^2$



(v_1, v_2, v_3) is

linearly dependent.

$a_1 = 2 \quad a_2 = 2 \quad a_3 = 1$

$$2v_1 + 2v_2 + 1 \cdot v_3 = 0$$

<u>Note:</u> can interpret this as saying $v_3 = (-2)v_1 + (-2)v_2$

i.e. $v_3 \in \mathrm{Span}(v_1, v_2)$

or $v_1 = \frac{1}{2}\left((-2)v_2 + (-1)v_3\right) = -v_2 - \frac{1}{2}v_3$

i.e. $v_1 \in \mathrm{Span}(v_2, v_3)$.

Note: A list $(v_1, \dots, v_n)$ of vectors in V is Linearly independent

$\iff$ $\operatorname{Span}(v_1) + \operatorname{Span}(v_2) + \cdots + \operatorname{Span}(v_n)$ is <u>DIRECT</u>.

Def^n: V is finite-dimensional when it can be spanned by a finite list of elements

i.e. $V = \operatorname{Span}(v_1, \dots, v_n)$.

(V infinite-dimensional if not).

e.g. $\mathbb{R}^n$ is finite-dimensional

because

$$\mathbb{R}^n = \operatorname{Span}(e_1, \dots, e_n, e_1, e_3, e_2+e_4)$$

$e_1 = (1, 0, \dots, 0)$

$e_2 = (0, 1, 0, \dots, 0)$

$\vdots$

$e_n = (0, 0, \dots, 0, 1)$

e.g. $\mathcal{P}(\mathbb{R})$ polynomials not finite dim.

(2.21) "∃ a redundant worker"

Lemma: If $(v_1, \dots, v_n)$ is linearly dependent,

then ① $\exists\ v_j$ in the span of the previous vectors on list,

i.e. $v_j \in \text{Span}(v_1, \dots, v_{j-1})$

② and we may remove v_j without changing span of list.

Proof: $(v_1, \dots, v_n)$ is lin. dep means $\exists\ (a_1, \dots, a_n) \in \mathbb{F}^n \setminus 0$

s.t. $a_1 v_1 + \dots + a_n v_n = 0$ (If 0 is on the list the list is lin. dep.)

Scan from end $a_n, a_{n-1}, a_{n-2}, \dots$ until we find $a_j \neq 0$

i.e. this eqn is actually $a_1 v_1 + \dots + a_j v_j = 0$ (nonzero number: a_j)

since $a_j \neq 0$ we can solve for v_j:

$$v_j = -a_j^{-1}(a_1 v_1 + \dots + a_{j-1} v_{j-1}).$$

exists since $\mathbb{F}$ field.

$\Rightarrow v_j \in \text{Span}(v_1, \dots, v_{j-1}) \Rightarrow$ prove ①

②: claim

$$\text{Span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n) = A$$

$$\text{Span}(v_1, \dots, v_n) = B$$

$A \subseteq B$ obvious

$B \subseteq A$: $w \in B$

$$B \ni w = b_1 v_1 + \dots + b_n v_n$$
$$= b_1 v_1 + \dots + b_{j-1} v_{j-1} + b_j\left(-a_j^{-1}(a_1 v_1 + \dots + a_{j-1} v_{j-1})\right)$$

$+ \; b_{j+1} v_{j+1} + \cdots + b_n v_n$.

$\in A$.

(2.23)

Thm: len (lin indep list) $\leq$ len (spanning list).

i.e. If $V = \text{span}(w_1, \ldots, w_n)$

and if $(u_1, \ldots, u_m)$ is lin indep.

then $m \leq n$.

Proof algorithm: use ordering + lemma:

① adjoin u_1 to spanning list:

$(u_1, w_1, \ldots, w_n)$ < still spans V; linearly dependent since $u_1 \in V = \text{Span}(w_1, \ldots, w_n)$

by the lemma, $\exists$ w_j which is in span $(u_1, w_1, \ldots, w_{j-1})$.

eliminate w_j, the result $(u_1, w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n)$

still spans V.

Continue this process: $(u_1, u_2, w_1 \cdots \hat{w_j} \cdots w_n)$ < still spans V; lin. dependent $\Rightarrow \exists\, w_\ell$ to eliminate

$\hat{w_j}$: w_j not in list.

each time add $u \overset{\text{Lemma}}{\Rightarrow} \exists$ vector on list linearly dependent on previous elements

this cannot be one of the u's by assumption. $\Rightarrow$ must be one of the w_i $\Rightarrow$ remove it.

so for each new u_i, there is a w_j which we can remove

and continue. $\Rightarrow$ $\# u_i \leq \# w_i$.

Review: $(v_1, \cdots, v_n)$ lin. dependent i.e. $\exists$ redundancy. from the p.o.v of span.

there is some nontrivial lin. comb. $= 0$.

i.e. can find $(a_1, \cdots, a_n)$ not all zero

st. $a_1 v_1 + \cdots + a_n v_n = 0$.

$a_1 v_1 + a_2 v_2 = 0$ some $a_i \neq 0$.

$2 \cdot v_1 - v_2 = 0$

Def: Let V be a finite dim. v.space (it has a finite spanning list i.e $V = \mathrm{Span}(v_1, \ldots, v_n)$ for some $v_i \in V$.),

A basis for V is a list $(v_1, \cdots, v_m)$ of vectors which

① spans V

② lin. independent.

Ex.: $V = \mathbb{F}^n$ has a natural basis as follows:

$$e_1 = (1, 0, 0, \cdots, 0).$$
$$e_2 = (0, 1, 0, \cdots, 0)$$
$$\vdots$$
$$e_n = (0, 0, \cdots, 0, 1).$$

① any vector in V can be written as

$v = (a_1, \ldots, a_n) \qquad a_i \in \mathbb{F}$

$v = a_1 \cdot e_1 + a_2 \cdot e_2 + \cdots + a_n \cdot e_n$

i.e. $\mathrm{Span}(e_1, \ldots, e_n) = V$.

② $(e_1, \cdots, e_n)$ is lin. indep.

If $\lambda_1 e_1 + \cdots + \lambda_n e_n = 0 \qquad \lambda_i \in \mathbb{F}$

$(\lambda_1, \ldots, \lambda_n) = 0 = (0, 0, \cdots, 0)$

$\Rightarrow \quad \lambda_i = 0 \quad \forall i$.

(linear comb is triv. no redundancy)

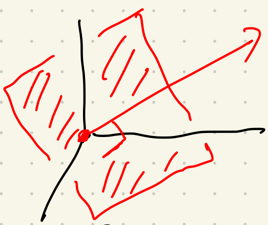
$\lambda_1(1,0,0) + \lambda_2(0,1,0) + \lambda_3(0,0,1)$

$= (\lambda_1, \lambda_2, \lambda_3)$

$= (0, 0, 0)$

$\mathbb{R}^3$

Ex.:



$U \subset \mathbb{R}^3$ lin subspace $\qquad U = \{(x, y, z) : x + y + z = 0\}$

Q.: find a basis for U?

(Pessimist approach)

Thm: Every finite dimensional v.space has a basis (highly non-unique choice.)

Pf: If $V = \{0\}$ then the empty list $(\ )$ is a basis.

Otherwise can express $V = \text{Span}(v_1, \dots, v_n)$ v_i nonzero.

for each $v_2, v_3, \dots, v_n$ we ask if it is in the span of previous vectors — if so, remove it
— if not keep it.

after this, span is unchanged, and no vector is in span of previous. By Lemma the resulting list is lin. indep.

$\Rightarrow$ lin indep. list spans V $\Rightarrow$ BASIS. $\square$.

Thm: (Build basis) (Optimist approach) Any lin. indep. list in a finite din. v.sp. can be extended to a basis

Pf: If $(v_1, \dots, v_m)$ is lin. independent, let $(w_1, \dots, w_n)$ be a spanning list.

<u>Step 1:</u> check if $w_1 \in \text{Span}\,\overbrace{(v_1, \dots, v_m)}^{B}$ If so, leave B unchanged. If not, add w_1 to B.

new $B = (v_1, \dots, v_m, w_1)$.

<u>Step j</u> If w_j in span of the latest B (from step j-1) leave unchanged, if not, add as before.

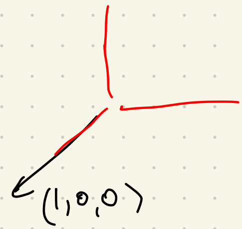
<u>finally after n steps:</u>
- have (by Lemma) a Lin indep. list.
- but $\underbrace{\text{Span}(w_1, \dots, w_n)}_{= V} \subseteq \text{Span}(v_1, \dots, v_m, w_{i_1}, \dots w_{i_k})$

list looks like

$(v_1, \dots, v_m, w_{i_1}, w_{i_2}, \dots, w_{i_k})$

$\mathbb{R}^3$ ( ) lin. indep

(x,y,z) $((1,0,0))$ lin. indep.



$x,y,z \in \mathbb{R}$

$$a(1,0,0) = 0$$

$$a = 0$$

$$\left(\overset{v_1}{(1,0,0)}, \overset{v_2}{(1,1,0)} \right)$$

$$a(1,0,0) + b(1,1,0) = (0,0,0)$$

$$\parallel$$

$$(a,0,0) + (b,b,0)$$

$$(a+b,\ b,\ 0) = (0,0,0)$$

1) $b = 0$

2) $a = 0$.

$$\left(\underline{\overset{v_1}{(1,0,0)}}, \underline{\overset{v_2}{(1,1,0)}}, \underline{\overset{v_3}{(1,1,1)}} \right) \ \underline{\text{Basis}}$$

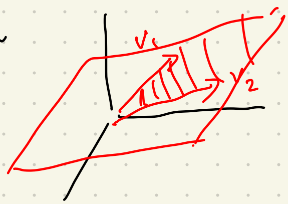
$$\text{Span} = \mathbb{R}^3$$

Defn (Incomplete): Let V be finite dimensional v.space. The dimension of V is the length of a basis for V.

This definition depends on the fact that all bases of V have the same length — otherwise "ill-defined".

Comments about explicit subspaces:

$\mathbb{F}^n$



Subspaces: $\mathrm{Span}(v_1, \dots, v_k)$

$$\begin{aligned} v_1 &= (a_1, \dots, a_n) & a_i \in \mathbb{F} \\ v_2 &= (b_1, \dots, b_n) & b_i \in \mathbb{F} \\ &\vdots \\ v_k &= (c_1, \dots, c_n) & c_i \in \mathbb{F} \end{aligned}$$

$k \times n$ grid of numbers

alternatively: $(a_1, \dots, a_n)$ $a_i \in \mathbb{F}$

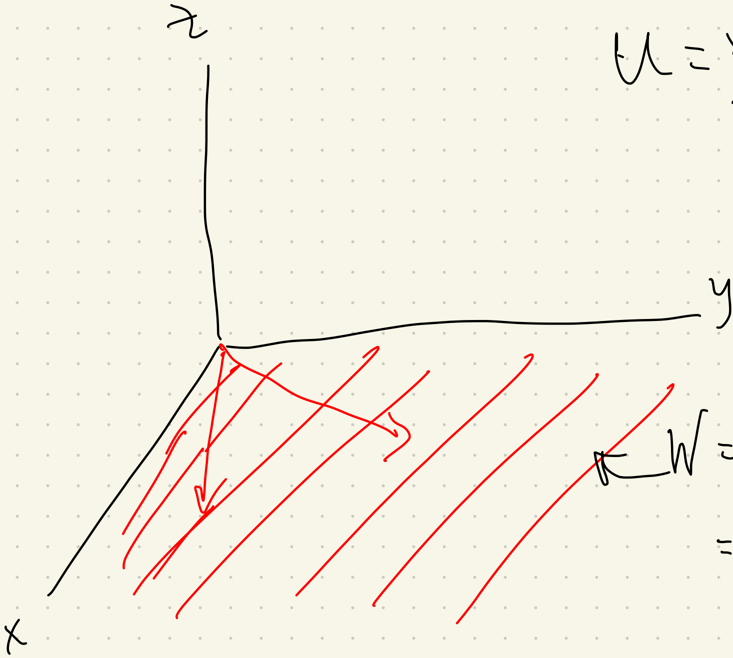
$(b_1, \dots, b_n)$ $b_i \in \mathbb{F}$

constraint

$$\left\{ (x_1, \dots, x_n) \in \mathbb{F}^n : \begin{array}{l} a_1x_1 + \dots + a_nx_n = 0 \\ b_1x_1 + \dots + b_nx_n = 0 \\ \vdots \end{array} \right\}$$

$U =$ xy plane $\subseteq \mathbb{R}^3$

subspace.



1. as a span

$W = \text{Span}((1,1,0), (1,2,0))$

$= \text{Span}((1,0,0), (0,1,0))$

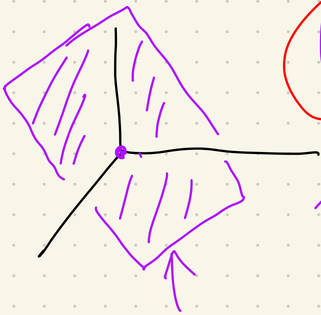
$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$

2. as a constraint

$\{ (x,y,z) : 0 \cdot x + 0 \cdot y + 1 \cdot z = 0 \}$ $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

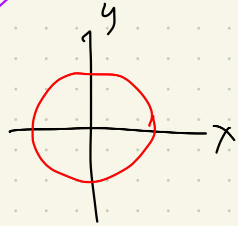
Solutions to this = linear subspace



$$U = \{(x,y,z) : \boxed{x+y+z=0}\},$$

Linear equations.

$$\{(x,y) : \cancel{x^2+y^2=1}\},$$



(1, 0, 0)
(0, 1, 0)
(0, 0, 1)

Q: find a basis for U

$(x,y,z) \in W \iff 0\cdot x + 0\cdot y + 1\cdot z = 0$

$$X = \left\{(x,y,z) : \begin{array}{c} 0\cdot x + 0\cdot y + 1\cdot z = 0 \\ \text{and} \\ 1\cdot x + 1y + 1\cdot z = 0 \end{array}\right\}$$

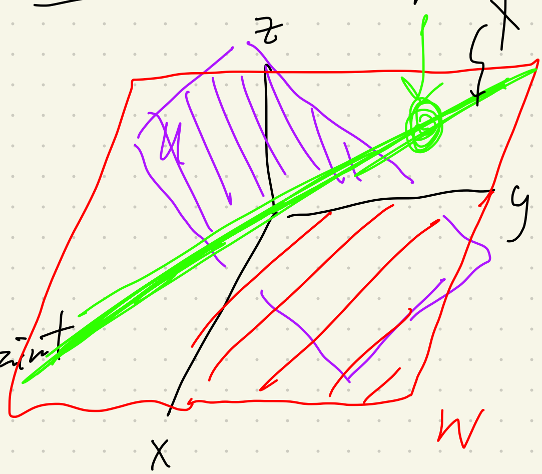
$(x,y,z) \in U \iff 1\cdot x + 1y + 1\cdot z = 0$

defines a linear subspace of $\mathbb{R}^3$

$\{(x,y,z)$ st. in W and in $U\}$.

$$\boxed{X = U \cap W}$$

1^{st} constraint 2^{nd} constraint



$\mathcal{P}_{\leq k}(\mathbb{F})$ polynomials of maximum degree k w/ coeffs in $\mathbb{F}$.

Q.: what is the dimension?

Basis: $(1, x, x^2, \ldots, x^k)$

$$\boxed{\dim(\mathcal{P}_{\leq k}(\mathbb{F})) = k+1}$$

$$\left\{\begin{array}{l} 1 \\ x+1 \\ x^2+x+1 \\ x^3+x^2+x+1 \\ \vdots \\ x^k + x^{k-1} + \cdots + x + 1 \end{array}\right.$$

$\mathcal{P}(\mathbb{F})$ v.space over $\mathbb{F}$

$(v_1, \ldots, v_m)$ $v_i \in \mathcal{P}(\mathbb{F})$. v_i has max degree d_i

s.t. $\mathcal{P}(\mathbb{F}) = \operatorname{Span}(v_1, \ldots, v_m)$.

$$a_1 v_1 + \cdots + a_m v_m.$$

$$v_1 = a_0 + a_1 x + a_2 x^2 + \cdots + a_{d_1} x^{d_1}$$

v_1 has a max degree d_1

$$\underbrace{\lambda_1}_{\mathbb{F}} v_1 = (\lambda_1 a_0) + (\lambda_1 a_1) x + \cdots + (\lambda_1 a_{d_1}) x^{d_1}$$

$$\lambda_1 v_1 + \cdots + \lambda_n v_n \qquad \vec{x} \longmapsto \vec{x} \cdot \vec{a}$$

---

linear constraints of the form $\boxed{a_1 x_1 + \cdots + a_n x_n = 0}$

$$\mathbb{F}^n \ni (x_1, \ldots, x_n) \qquad a_i \in \mathbb{F}$$

this constraint is really $f^{-1}(0)$

$$f : \mathbb{F}^n \longrightarrow \mathbb{F}$$

$$(x_1, \ldots, x_n) \longmapsto a_1 x_1 + \cdots + a_n x_n$$

Span $\longrightarrow$ Finite dimensionality.

Linear (In)dependence $\longrightarrow$ basis (lin ind + spanning)

Lemma: length (linearly indep list) $\leq$ length (spanning list)

$\Rightarrow$ Def^n of Dimension

Def: $\operatorname{Dim} V = \operatorname{len}(\text{basis})$ — V has a basis! (Lemma ✓)

this makes sense only if all bases have same length!

Lemma: If B, B' are bases for $V \Rightarrow \operatorname{len}(B) = \operatorname{len}(B')$

Pf: B spans, B' lin indep $\underset{\text{Lemma}}{\Rightarrow} \operatorname{len}(B') \leq \operatorname{len}(B)$

B' spans, B lin indep $\underset{\text{Lemma}}{\Rightarrow} \operatorname{len}(B) \leq \operatorname{len}(B')$

$\Rightarrow \operatorname{len}(B) = \operatorname{len}(B')$.

ex: $\mathcal{P}_{\leq n}(\mathbb{F}) = \{ a_0 + a_1 x + \cdots + a_n x^n : a_i \in \mathbb{F} \}$.
has a basis $B = (p_0, p_1, \ldots, p_n)$

$p_0 = 1$
$p_1 = x$
$p_2 = x^2$
$\vdots$
$p_n = x^n$

Basis

spans: $= a_0 p_0 + a_1 p_1 + \cdots + a_n p_n$.

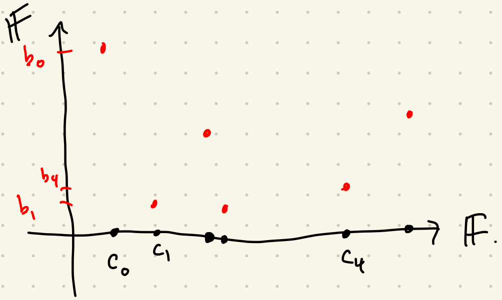
lin. indep: $a_0 + a_1 x + \cdots + a_n x^n = 0$

$\Leftrightarrow a_0 = a_1 = a_2 = \cdots = a_n = 0$

$\operatorname{len} B = n+1$ $\quad \boxed{\dim \mathcal{P}_{\leq n}(\mathbb{F}) = n+1}$.

Ex.: $\mathcal{P}_n(\mathbb{F})$ same vector space, different basis.

$\dim \mathcal{P}_n(\mathbb{F}) = n+1.$



Problem: given input numbers $c_0, c_1, \dots, c_n \in \mathbb{F}$ and output values $b_0, \dots, b_n$, Can we find a polynomial which fits the data. i.e. p s.t. $p(c_i) = b_i$?

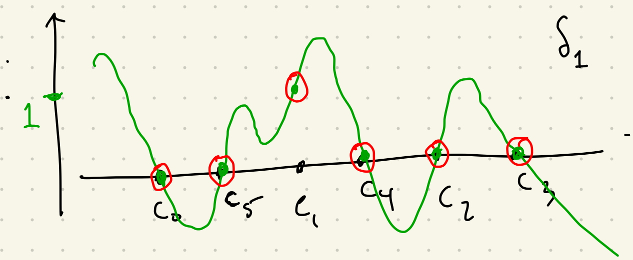
heuristic.

Guess: this might be possible with $\mathcal{P}_n(\mathbb{F})$ since $\dim = n+1$.

Reason why problem is hard: basis we are used to is $(1, x, x^2, \dots, x^n)$

so unclear how choice of coeffs $a_0 + a_1 x + \dots + a_n x^n$ affects value of poly at $c_0, \dots, c_n$.

alternatively: Suppose we had a different basis $(\delta_0, \dots, \delta_n)$ adapted to positions $c_0, \dots, c_n$ as follows.



$\delta_1(c_0) = 0$

$\delta_1(c_1) = 1$

$\delta_1(c_2) = 0$

$\vdots$

$\delta_1(c_n) = 0.$

have "indicator basis" adapted to $(c_0, \dots, c_n)$

$$\delta_i(c_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else.} \end{cases}$$

Can we solve using this basis?

need p s.t. $p(c_i) = b_i$

$b_i \delta_i$ has values

0	at	c_0
0	at	c_1
0	at	c_{i-1}
b_i	at	c_i
0		c_{i+1}
$\vdots$		
0		c_n.

conclude $b_0\delta_0 + \dots + b_n\delta_n = p$!!

satisfies all constraints!

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how do we find such a basis? (Lagrange Interpolation) data fitting.

δ_i should have value 1 at c_i and 0 at c_j $j \neq i$

$$\boxed{\delta_i = \frac{(x-c_0)(x-c_1)\cdots(x-c_{i-1})(x-c_{i+1})\cdots(x-c_n)}{(c_i-c_0)(c_i-c_1)\cdots(c_i-c_{i-1})(c_i-c_{i+1})\cdots(c_i-c_n)}}$$

has zeros at all c_j $j \neq i$.

at c_i it has value $(c_i-c_0)(c_i-c_1)\dots$

this will work if the c_i are pairwise distinct

Summary: Lagrange Interpolation

Fix pairwise distinct elts of $\mathbb{F}$

$$c = (c_0, \dots, c_n)$$

Create basis $B_c = (\delta_0, \dots, \delta_n)$

s.t. $\delta_i(c_j) = \begin{cases} 1 & i=j \\ 0 & \text{else.} \end{cases}$

$$\delta_i = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-c_j)}{(c_i-c_j)}$$

Fit to data is $\sum_{i=0}^{n} b_i \delta_i = p. \in \mathcal{P}_n(\mathbb{F})$

$$\boxed{p(c_i) = b_i}$$

Rem: How do we know it's a basis?

It is lin indep.
Its length $\boxed{\text{is } n+1}$
$\Downarrow$
basis

It is linearly independent!:

If $(a_0, \dots, a_n)$ are such that

$$f = a_0\delta_0 + \dots + a_n\delta_n = 0 \quad (*)$$

want to show all $a_i = 0$ $\implies$ list B_c lin indep.

but $(*) \Rightarrow \boxed{f(c_i) \overset{\forall i}{=} a_i = 0}$

$\square$.