Uses of Gaussian Elimination

Start: list of vectors in V $(v_1, \dots, v_k)$ and a basis $(e_1, \dots, e_n)$

numerical data
$$v_1 = a_{11} e_1 + a_{12} e_2 + \cdots + a_{1n} e_n$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$v_k = a_{k1} e_1 + a_{k2} e_2 + \cdots + a_{kn} e_n$$

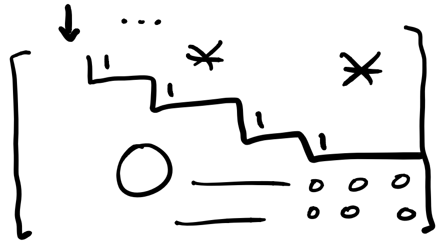
allowed ERO

- Switch
- Scale
- Shear

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} \quad k \times n \text{ matrix}$$

$$\downarrow \text{ERO}_1$$
$$\downarrow \text{ERO}_2$$
$$\downarrow \cdots$$

output: $k \times n$ matrix in RE / RRE



lin. indep.

guaranteed: output represents a list with same span.
$(w_1, w_2, \dots, 0, 0)$

Rmk: ① significance of zero rows in result is that there are redundancies in original list. i.e. original list lin. dep.

② the nonzero rows in result are automatically linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

RE form.

corresponding list is (w_1, w_2, w_3)

$w_1 = 1 \cdot e_1 + 2 \cdot e_2 + 3e_3 + 4e_4$

$w_2 = 1e_3 + 6e_4$

$w_3 = 0$

show (w_1, w_2) lin. indep.

$$a w_1 + b w_2 = 0$$

wts $a = 0$ and $b = 0$

coeff of e_1 is $a = 0$

coeff of e_3 is $3a + b = 0$

$$\Downarrow$$
$$\underline{a=0 \text{ and } b=0}$$

$$w = 1e_i + c \cdot e_j + \cdots$$
$$w' = 0 \cdot e_i + 1e_j + \cdots$$

$aw + bw'$ has e_i coeff a

e_j coeff $ac + b$

$$aw + bw' = 0 \Rightarrow \underset{=0}{\text{coeffs in basis}}$$

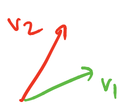
$$\Rightarrow \begin{matrix} a = 0 \\ \cancel{ac} + b = 0 \end{matrix} \Rightarrow \begin{matrix} a=0 \\ b=0 \end{matrix}$$

Conclusion: the nonzero rows of RE / RRE represent a **basis** for the span of the original list.

2x2 matrix.

Example: $v_1 = a e_1 + b e_2$

$v_2 = c e_1 + d e_2$ $\Big\}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$



Under what condition is (v_1, v_2) lin dependent.

1) if $a \neq 0$ use it, no switch needed.
2) Scale $R_1 \to a^{-1} R_1$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 1 & a^{-1}b \\ c & d \end{bmatrix}$$

3) Shear c away

$R_2 \to R_2 - c \cdot R_1$

$$\begin{bmatrix} 1 & a^{-1}b \\ 0 & d - a^{-1}bc \end{bmatrix}$$

Inspect

a) $d - a^{-1}bc \neq 0$

$\Downarrow$

scale 2^{nd} row

get $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$

lin indep.

b) $d - a^{-1}bc = 0$

RE with a 0 row

$\Downarrow$

If $a \neq 0$ cond for lin dep is $ad - bc = 0$ lin. dependent!

If $a = 0$ want to use c if $c = 0 \Rightarrow$ lin dep!!

so lin dep if $a = 0$ and $c = 0$ taken care of by

and $c \neq 0$ use c as early bird.

switch $\begin{bmatrix} c & d \\ a & b \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 1 & c^{-1}d \\ a & b \end{bmatrix}$

$\xrightarrow{\text{shear}} \begin{bmatrix} 1 & c^{-1}d \\ 0 & b - c^{-1}ad \end{bmatrix}$ — lin dep if $b - c^{-1}ad = 0$

$\Updownarrow$

$bc - ad = 0$

all roads lead to same eqn:

$$\boxed{(v_1, v_2) \text{ lin dep} \Leftrightarrow ad - bc = 0}$$

"$ad - bc$ determines if list lin. dep. or not"

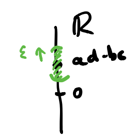
DETERMINANT of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

note: If $ad - bc \neq 0$

i.e. lin independence holds,

then if we perturb matrix a small amount $\underbrace{ad - bc}_{\text{continuous function of } (a,b,c,d)}$ will remain $\neq 0$.

OTOH If we have lin dep vectors, a small modif may make them indep.



Linear Independence is an "OPEN" condition

Solving linear equations   Find all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

satisfying $\quad f_1(x) = f_2(x) = \cdots = f_k(x) = 0$

f_i are linear functions
$$\begin{aligned} f_1 &= a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \\ f_2 &= a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \\ &\vdots \\ f_k &= a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{kn} x_n \end{aligned}$$

Note: this is a special case of previous skill, since the linear functions $\mathbb{F}^n \xrightarrow{f} \mathbb{F}$ form a vector space with basis $(e_1, \dots, e_n)$:
$$\begin{aligned} e_1(x_1, \dots, x_n) &= x_1 \\ e_2(x_1, \dots, x_n) &= x_2 \\ &\vdots \\ e_n(x_1, \dots, x_n) &= x_n \end{aligned}$$

the linear system consists of $(f_1, \dots, f_n)$ where
$$\begin{aligned} f_1 &= a_{11} e_1 + \cdots + a_{1n} e_n \\ &\vdots \\ f_k &= a_{k1} e_1 + \cdots + a_{kn} e_n \end{aligned}$$

will apply GE to $(f_1, \dots, f_k)$ to produce a new linear system $(g_1, \dots, g_k)$ and then solve

$\{x : g_1(x) = g_2(x) = \cdots = g_k(x) = 0\}$.

ERO $\begin{matrix} f_1 \\ f_2 \end{matrix}$ $\qquad f_i \mapsto f_i + \lambda f_j$

$f_i \mapsto \lambda f_i$

a point x which was in common zero set of $(f_1, \dots, f_k)$ will not. be in zero set of $(g_1, \dots, g_k)$.

same arg. holds in reverse, showing

$$\text{Zeros}(f_1, \dots, f_k) = \text{Zeros}(g_1, \dots, g_k)$$

$$\text{Sol. set} = \{ t(-2, 1, 0, 0) : t \in \mathbb{F} \}$$

$(1, 2, 5, 7) \in \mathbb{R}^4$

Ex.: $f_1 = x_1 + 2x_2 + x_3 + x_4 = 0$

$f_2 = 2x_1 + 4x_2 + 5x_3 + 3x_4 = 0$

$f_3 = 4x_1 + 8x_2 + 7x_3 + 4x_4 = 0$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 5 & 3 \\ 4 & 8 & 7 & 4 \end{bmatrix}$$

$\mathbb{R}^4 \ni (x_1, x_2, x_3, x_4)$

$R_2 \longmapsto R_2 - 2R_1$

$$\begin{matrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 4 & 8 & 7 & 4 \end{matrix}$$

$R_3 \longmapsto R_3 - 4R_1$

$$\begin{matrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 3 & 0 \end{matrix}$$

$R_2 \longleftrightarrow R_3$

f_1 is a linear f^n on $\mathbb{R}^4$

The set of linear f^{ns} on $\mathbb{R}^4$ is a vector space!

The Dual Space. (of $\mathbb{R}^n$).

$$\begin{matrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 1 \end{matrix} \xrightarrow{\frac{1}{3}R_2} \begin{matrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

RE form

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \mapsto R_1 - R_3]{R_1 \mapsto R_1 - R_2} \begin{matrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

RRE $\square$

<u>new system</u>

$$g_1 = x_1 + 2x_2 + x_3 + x_4$$

$$g_2 = x_3$$

$$g_3 = x_4$$

$$g_1 = g_2 = g_3 = 0$$

$$x_4 = 0$$
$$x_3 = 0$$
$$x_1 + 2x_2 = 0$$

no x_2 echelon $\Rightarrow$ cannot solve for x_2!

Let $x_2 = t$ a param.

$$x_1 = -2t$$
$$x_2 = t$$
$$x_3 = 0$$
$$x_4 = 0$$

Sol. set $= \{ t(-2, 1, 0, 0) : t \in \mathbb{F} \}$

How do we use this technique to solve

$$\left(\begin{array}{c} f_1(x) = c_1 \\ f_2(x) = c_2 \\ \vdots \\ f_k(x) = c_k \end{array} \right)$$

$\mathbb{R}^4$

f linear f^n on $\mathbb{R}^4$

$$x_1(1,0,0,0) + x_2(0,1,0,0) + x_3(0,0,1,0) + x_4(0,0,0,1)$$

$\|$ using the def^n of linearity.

$$f(\overbrace{(x_1, x_2, x_3, x_4)}) =$$

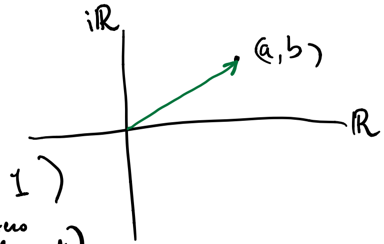
$$x_1 \underbrace{f(1,0,0,0)}_{a_1} + x_2 \underbrace{f(0,1,0,0)}_{a_2} + x_3 \underbrace{f(0,0,1,0)}_{a_3} + x_4 \underbrace{f(0,0,0,1)}_{a_4}$$

$$f = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$$

X_1

$(1, 2, 3, 4)$

① $\mathbb{C} = \{ a + ib \ , \quad a,b \in \mathbb{R} \quad i^2 = -1 \}$.



this is a field

is also a 1-d v.sp over $\mathbb{C}$ (with basis 1)
(or any nonzero element).

"the complex line"?

② $\mathbb{R}$ field, 1-d vsp$/\mathbb{R}$.

$\mathbb{R}^2$ 2d v.space over $\mathbb{R}$.

③ $\mathbb{R} \subset \mathbb{C}$, as a result

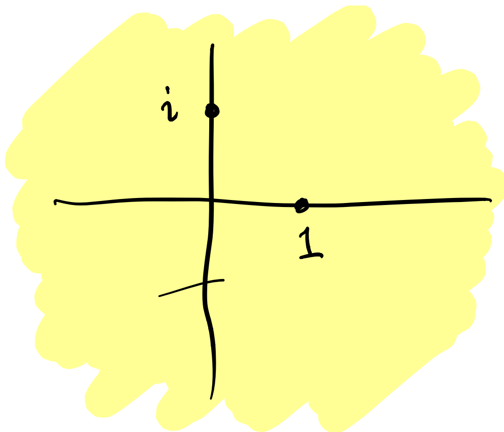
$\mathbb{C}$ is a v.space over $\mathbb{R}$.

$$\left(\lambda \in \mathbb{R} \ \cdot \ a+ib \in \mathbb{C} \right) = \lambda \cdot (a+ib)$$
$$= \lambda a + i \lambda b.$$

(forget about $\mathbb{C} \times \mathbb{C} \xrightarrow{m} \mathbb{C}$
only remember $\mathbb{R} \times \mathbb{C} \to \mathbb{C}$).

<u>Q.</u>: what is the dimension of $\mathbb{C}$ as a $\mathbb{R}$-vector space?

<u>basis for $\mathbb{C}$ over $\mathbb{R}$:</u> $(1, i)$



any cx number

is $a e_1 + b e_2$

$e_1 = 1$ $\quad a, b \in \mathbb{R}$.
$e_2 = i$

this 2d $\mathbb{R}$ v.space "$\mathbb{C}$" is in bijection with our favourite 2d $\mathbb{R}$ v.space $\mathbb{R}^2$

by sending $a + ib \longmapsto (a, b)$.