Morphisms between algebraic structures

e.g. G group = set with oper $m: G \times G \to G$
with identity e
with inverses.

$G = (G, m_G, e_G, inv_G)$ $H = (H, m_H, e_H, inv_H)$

A $\begin{pmatrix} \text{group} \\ \text{homomorphism} \end{pmatrix}$ $G \xrightarrow{f} H$ is a map of set

$$g_1, g_2 \in G \quad \left\{ \begin{array}{l} f(m_G(g_1, g_2)) = m_H(f(g_1), f(g_2)) \\ f(e_G) = e_H \\ f(inv(g)) = inv(f(g)). \end{array} \right.$$

A Category is what we get when considering
objects and Morphisms between objects

(Sets, maps of sets) = Category of Sets
(Groups, homomorphism) = Category of Groups.
(Vector spaces, Linear maps) = Category of vector spaces

Def^n: Let U and V be vector spaces over same field $\mathbb{F}$.
A linear map $L: U \to V$
is a map of sets preserving str, i.e.

$$\left\{ \begin{array}{l} L(\lambda \cdot u_1 + u_2) \\ = \lambda L(u_1) + L(u_2) \end{array} \right\} \forall u_1, u_2 \ \forall \lambda \iff \left\{ \begin{array}{ll} L(u_1 \underset{U}{+} u_2) = L(u_1) \underset{V}{+} L(u_2). & \forall u, u_1, u_2 \in U \\ L(\lambda \underset{U}{\cdot} u) = \lambda \underset{V}{\cdot} L(u) & \forall \lambda \in \mathbb{F} \end{array} \right.$$

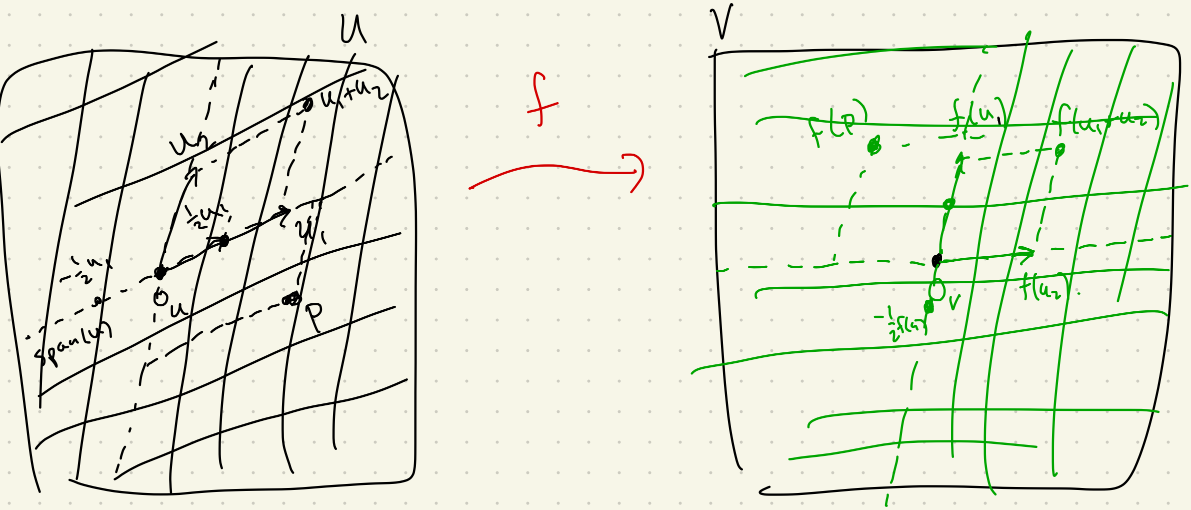
e.g. automatic ① $L(0_u) = 0_v$. $L(0+0) = L(0) + L(0)$

$ \| $

$L(0) \Rightarrow L(0) = 0.$

② $L(-u) = L(-1 \cdot u)$

$= -1 \cdot L(u) = -L(u).$



Data needed to define f:

1. $u_1 \mapsto f(u_1)$.
2. u_2 lin. indep. from u_1

   $u_2 \mapsto f(u_2)$

Knowing *only* where u_1, u_2 are sent,

we know where any vector $v \in \mathrm{Span}(u_1, u_2)$ is sent.

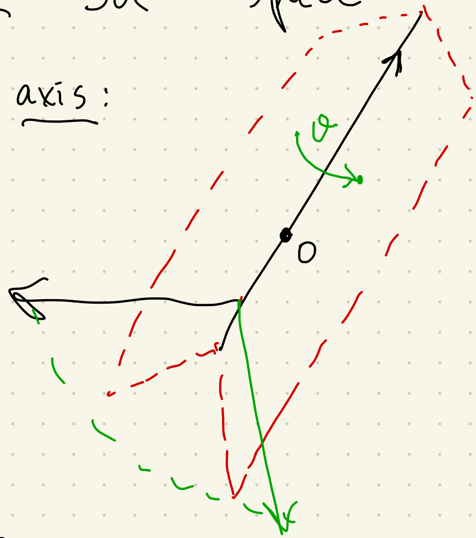
If $B = (e_1, \dots, e_n)$ is a basis for U

and we know the values $f(e_1), \dots, f(e_n)$

then we know all possible values!

e.g.: 3d space rotations.

axis:



- line in 3d. ℓ through O.
- angle $\theta \in [0, 2\pi)$
- Rotation $R(\ell, \theta)$

$$R(u_1 + u_2) = R(u_1) + R(u_2)$$

$$R(\lambda u) = \lambda R(u).$$

Rotations are linear maps $R: \mathbb{R}^3 \to \mathbb{R}^3$

Ex.: Factory: can produce several products $(P_1, \dots, P_n)$ (recipes)

out of various materials or ingredients.
$(Q_1, \dots, Q_K)$ (ingredients)

roughly "linear" map from $\mathbb{R}^n = \text{Span(Products)}$

to $\mathbb{R}^k = \text{Span(ingredients)}$

$P_1 \leftrightarrow (1, 0, \dots, 0)$
$P_2 \leftrightarrow (0, 1, \dots, 0)$
$P_n \leftrightarrow (0, \dots, 0, 1)$

$Q_1 = (1, 0, \dots, 0)$
$\vdots$
$Q_k = (0, \dots, 0, 1)$

$$\mathbb{R}^n \xrightarrow{F} \mathbb{R}^k$$

$(1, 3)$

Explicit description of a linear map using matrices.

① $L: \mathbb{R}^n \to \mathbb{R}^k$

to know L completely, only need its value on a basis,
use standard basis $\mathcal{B} = ((1,0,\dots,0), \dots, (0,0,\dots,0,1))$ of len $= n$.

e_1 e_n

Info: $[\ L(e_1) \quad L(e_2) \quad L(e_3) \ \dots \quad L(e_n)\ ]$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{1n} \\ a_{21} & a_{22} & a_{23} & & \vdots \\ a_{31} & a_{32} & \vdots & & \vdots \\ \vdots & \vdots & & & \\ a_{k1} & a_{k2} & a_{k3} & & a_{kn} \end{bmatrix}$$

$L(e_1) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{k1} \end{pmatrix}$ vector in $\mathbb{R}^k$

$k \times n$ matrix representing L, in the standard basis of $\mathbb{R}^n$ and the standard basis of $\mathbb{R}^k$.

$A = [a_{ij}] \quad a_{ij} \in \mathbb{R}.$

Now that we know value of L on basis, we can determine all values.

if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ general vector,

$$L(x = x_1 e_1 + \dots + x_n e_n) = x_1 L(e_1) + \dots + x_n L(e_n).$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ \vdots \\ a_{kn} \end{pmatrix} \in \mathbb{R}^k$$

$$
\underbrace{
\begin{array}{c c}
 & \overleftrightarrow{\quad n \quad} \\
k \updownarrow &
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & \vdots & & \vdots \\
\vdots & \vdots & & \vdots \\
a_{k1} & a_{k2} & \cdots & a_{kn}
\end{bmatrix}
\end{array}
\begin{array}{c c}
n \updownarrow &
\begin{bmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{bmatrix}
\end{array}
}_{\text{lin. comb. of columns of } A \text{ w/ coeffs } x_1, \dots, x_n}
=
\begin{bmatrix}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\vdots \\
a_{k1}x_1 + \cdots + a_{kn}x_n
\end{bmatrix}
=
\begin{pmatrix}
a_{1\bullet} \cdot x \\
a_{2\bullet} \cdot x \\
\vdots \\
a_{k\bullet} \cdot x
\end{pmatrix}
\in \mathbb{R}^k
$$

product of $k \times n$ matrix and a $n \times 1$ matrix (column vector) to give $k \times 1$ matrix (column vector),

alternate form

$$
= \begin{pmatrix}
\sum_{j=1}^{n} a_{1j} x_j \\
\sum_{j=1}^{n} a_{2j} x_j \\
\vdots \\
\sum_{j=1}^{n} a_{kj} x_j
\end{pmatrix}
\in \mathbb{R}^k.
$$

② U, V are v.sp. over same field $\mathbb{F}$.

$L : U \to V$ linear map.

want explicit repres. of L.

Need to make some choices

① choose a basis $B_u = (e_1, \cdots, e_n)$ for U.

② choose a basis $B_V = (f_1, \cdots, f_k)$ of V.

with these two choices, we get a matrix of L

$$M(L, B_u, B_V) = \begin{array}{cc} & \overset{n}{\longleftrightarrow} \\ k \updownarrow & \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix} \end{array} = {}_{B_V}L_{B_u}$$

Alternate notation

$L(e_j) = a_{1j} f_1 + \cdots a_{kj} f_k$

$L(e_j)$ goes in j^{th} col : $L(e_j) = \sum_{i=1}^{k} a_{ij} f_i$

Factory producing cars $(C_1, C_2 \cdots C_n)$.

w/ ingredients $(M_1 \ldots M_k)$

	C_1	C_2	C_3	C_4
M_1: Metal$_1$	10	5	6	⋮
M_2 = Metal$_2$	3	1	3	
M_3 = Rubber	2	1	2	
M_4 = Plastic	17	0	7	

matrix of the linear map from products to ingredients.

Ex.: $\mathcal{P}_2(\mathbb{R}) \xrightarrow[D = \frac{d}{dx}]{} \mathcal{P}_1(\mathbb{R})$

$B = (1, x, x^2)$ $\qquad$ $(1, x) = B'$

$f_1, f_2 \in \mathcal{P}_2(\mathbb{R})$

$c \in \mathbb{R}$.

$$\frac{d}{dx}(f_1 + c \cdot f_2) = \frac{d}{dx} f_1 + c \cdot \frac{d}{dx} f_2$$

$$D(u_1 + \lambda u_2) = D(u_1) + \lambda \cdot D(u_2)$$

D linear map! . It has a matrix!

$D(1)=0 \qquad D(x)=1 \qquad D(x^2)=2x$

$$ {}_{B'}D_{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} $$

$B' = (1, x)$

$$ p = a\,1 + b \cdot x + c\,x^2 $$

$$ Dp = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \end{bmatrix} $$

Inspect

$$ Dp = b \cdot 1 + 2c \cdot x $$

Composition of linear maps / Matrix multiplication.

cof / bookkeeping

V, W f.d v.spaces over $\mathbb{F}$ $\qquad T: V \to W$ $\quad \ni \; T(v_5) = ?$

$= * w_1 + \cdots + * w_k$

I matrix of linear map T

in bases $\begin{cases} \beta = (v_1, \cdots, v_n) & \text{for } V \\ \gamma = (w_1, \cdots, w_k) & \text{for } W \end{cases}$

$$\left({}_\gamma[T]_\beta\right) = {}_\gamma T_\beta = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix},$$

$\shortparallel$

$M(T, \beta, \gamma)$

$a_{ij} \in \mathbb{F}$

where columns show coeffs of $T(v_j)$ in the basis γ

i.e.
$$T(v_j) = a_{1j} w_1 + \cdots + a_{kj} w_k = \sum_{i=1}^{k} a_{ij} w_i$$

II matrix of a vector $x \in V$ in basis β is

$${}_\beta x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{where} \quad x = x_1 v_1 + \cdots + x_n v_n \qquad x_i \in \mathbb{F}$$

$= M(x, (1), \beta)$

we may view any vector $x \in V$ as defining a linear map $\mathbb{F} \xrightarrow{x} V$

$$1 \longmapsto x$$
$$\lambda \longmapsto \lambda x$$

std basis elt. of $\mathbb{F}$

III Apply linear map to the vector:

${}_\gamma(T(x)) = ?$ what explicitly is the result $T(x)$?

$$T(x) = T(x_1 v_1 + \cdots + x_n v_n) = x_1 T(v_1) + \cdots + x_n T(v_n)$$
$$= x_1 \left(\sum_{i=1}^{k} a_{i1} w_i \right) + \cdots + x_n \left(\sum_{i=1}^{k} a_{in} w_i \right)$$

$${}_\gamma(T(x)) = \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix} \qquad \begin{matrix} y_1 = x_1 a_{11} + \cdots + x_n a_{1n} \\ \vdots \\ y_k = x_1 a_{k1} + \cdots + x_n a_{kn} \end{matrix} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{k1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{k2} \end{bmatrix} + \cdots$$

columns of ${}_\gamma T_\beta$

$$= r\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ y_r \\ \vdots \\ a_{k1}x_1 + \cdots + a_{kn}x_n \end{bmatrix}$$

r^{th} entry

$(k \times n) \cdot (n \times 1)$

Abstract:

$$\begin{array}{ccc} V & & W \\ \cup & & \cup \\ x & \longmapsto & T(x) \end{array}$$

Explicit:

$$_\beta[x] \longmapsto {}_\gamma[T]_\beta \cdot {}_\beta[x] = {}_\gamma[T(x)]$$

$(k \times n) \quad (n \times 1) \qquad k \times 1$

IV Composition of Linear maps (full matrix multip.)

U, V, W f.d.v.sp$/\mathbb{F}$

$$\begin{tikzcd} \underset{\alpha}{U} \arrow[r, "S"] \arrow[rr, bend right, "T \circ S"'] & \underset{\beta}{V} \arrow[r, "T"] & \underset{\gamma}{W} \end{tikzcd}$$

Bases

Lemma: If S, T are linear maps, then $T \circ S$ is too

$$\begin{aligned} TS(\lambda u_1 + u_2) &\overset{\text{Lin of } S}{=} T(\lambda S(u_1) + S(u_2)) \\ &\overset{\text{Lin of } T}{=} \lambda\, T(S(u_1)) + T(S(u_2)) \\ &\overset{\text{Def of } \circ}{=} \lambda\, (T\circ S)(u_1) + (T \circ S)(u_2). \quad \square \end{aligned}$$

Q.: If $\alpha = (u_1, \cdots, u_m)$, $\beta = (v_1, \cdots, v_n)$, $\gamma = (w_1, \cdots, w_k)$ are bases for U, V, W,

and if ${}_{\beta}S_{\alpha}$ and ${}_{\gamma}T_{\beta}$ are matrices of S & T.

$${}_{\beta}S_{\alpha} = \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \qquad {}_{\gamma}T_{\beta} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix}$$

Then what is the matrix of TS $(T \circ S)$?

i.e. what is ${}_{\gamma}(TS)_{\alpha} = \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \vdots \\ c_{k1} & \cdots & c_{km} \end{bmatrix}$?? what is c_{ij}

Simply Compute :

$\gamma(TS)_\alpha$ is a $k \times m$ matrix

$$
\begin{array}{cc}
 & \overleftrightarrow{\quad m \quad} \\
k \updownarrow &
\begin{bmatrix}
c_{11} & \cdots & c_{1m} \\
\vdots & & \vdots \\
c_{k1} & \cdots & c_{km}
\end{bmatrix}
\end{array}
$$

j^{th} column is expansion of $(TS)(u_j)$

in γ basis

$$(TS)(u_j) = * \, w_1 + * \, w_2 \cdots + * \, w_k$$

$$= T\left(S(u_j) = \sum_{p=1}^{n} \overbrace{b_{pj}}^{j^{th} \text{ column of } [S]} v_p \right)$$

$$= \sum_{p=1}^{n} b_{pj} \, T(v_p)$$

$$= \sum_{p=1}^{n} b_{pj} \left(\sum_{i=1}^{k} a_{ip} w_i \right) = \sum_{i=1}^{k} \underbrace{\left(\sum_{p=1}^{n} a_{ip} b_{pj} \right)}_{c_{ij}} w_i$$

$$
\begin{array}{c c}
 & \\
i & \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix}
\end{array}
\begin{array}{c}
j \\
\begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}
\end{array}
=
\begin{array}{c c}
 & j \\
i & \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & c_{ij} & \vdots \\ c_{k1} & \cdots & c_{km} \end{bmatrix}
\end{array}
$$

$k \times n$ $\qquad$ $n \times m$ $\qquad$ $k \times m$

$$a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + a_{i3} b_{3j} + \cdots + a_{in} b_{nj} = c_{ij}$$

<u>Def</u>^{n} of matrix multiplication

with this def^n, it is true that

compos. $\qquad$ matrix mult.

$$_\gamma(T \circ S)_\alpha \overset{\star}{=} {}_\gamma T_\beta \cdot {}_\beta S_\alpha$$

$$_\gamma[TS(x)] \overset{\star}{=} {}_\gamma(TS)_\alpha \cdot {}_\alpha x \overset{\star}{=} \Big(\overbrace{({}_\gamma T_\beta \cdot {}_\beta S_\alpha)}^{\text{matrix - vect}} \cdot {}_\alpha x\Big)$$

(3) $\qquad$ (★) $\qquad$ (2) $\qquad$ (1) ass.

abstr $\qquad$ || ass. of $\circ$ $\qquad$ Exp $\qquad$ ||

$$_\gamma(T(S(x))) = {}_\gamma T_\beta \, {}_\beta(Sx) = {}_\gamma T_\beta \cdot ({}_\beta S_\alpha \cdot {}_\alpha x)$$

(4) $\qquad$ (5) $\qquad$ (6)

1. change of basis
2. Isom.
3. Inverses + Books

V

What is the dependence of the matrix $M(T, \beta, \gamma)$ on the choice of bases β, γ?

$$T : V \longrightarrow W$$

$$\beta = (v_1, \dots, v_n) \qquad \gamma = (w_1, \dots, w_k).$$

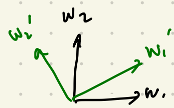
$$\begin{bmatrix} & T v_j & \end{bmatrix}$$

$${}_\gamma[T]_\beta = [c_{ij}] \quad \text{where} \quad T v_j = \sum_{i=1}^{k} c_{ij} w_i$$

Suppose we change the choices: $\beta' = (v_1', \dots, v_n')$ is another basis for V

$\gamma' = (w_1', \dots, w_k')$ .... W.

We want to express ${}_{\gamma'}[T]_{\beta'}$ in terms of ${}_\gamma[T]_\beta$



$$\underset{\beta'}{V} \xrightarrow{I_V} \underset{\beta}{V} \xrightarrow{T} \underset{\gamma}{W} \xrightarrow[\text{[I]}]{\text{identity } I_W} W_{\gamma'}$$

① $\underset{\gamma}{W} \xrightarrow{I} \underset{\gamma}{W}$

Identity matrix

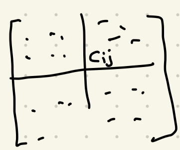
$${}_\gamma[I]_\gamma = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \, 1 \end{bmatrix} \qquad \begin{bmatrix} & I w_j & \end{bmatrix}$$

$\gamma = (w_1, \dots, w_k)$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

$$I w_j = w_j$$

② $\underset{\gamma}{W} \xrightarrow{I} \underset{\gamma'}{W}$



j^{th} column of ${}_{\gamma'}[I]_\gamma$

= coeffs of

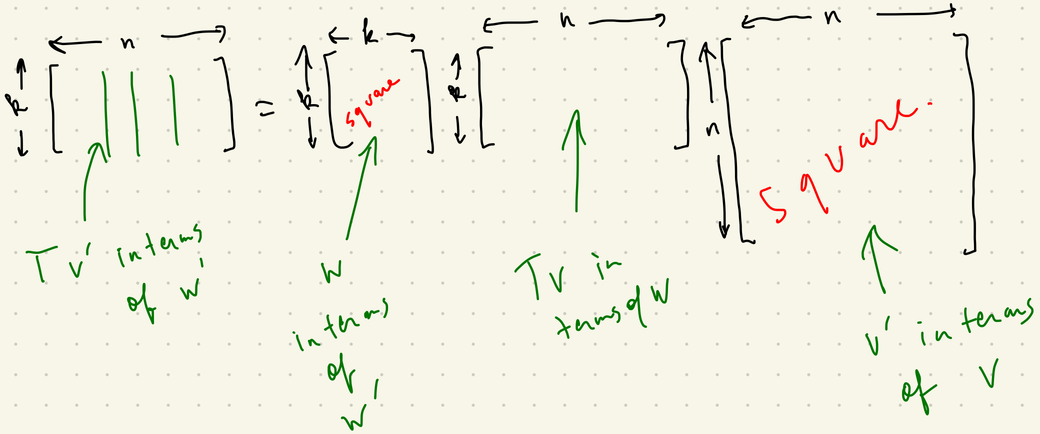
$$w_j = I w_j = c_{1j} w_1' + \cdots + c_{kj} w_k'$$

$V \to W$ $\quad$ $\dim V = n$ $\quad$ $\dim W = k$

To find matrix of T in new bases β', γ',

$${}_{\gamma'}[T]_{\beta'} = {}_{\gamma'}[I_W \, T \, I_V]_{\beta'} \qquad \underset{\beta'}{V} \xrightarrow{I} \underset{\beta}{V} \to \underset{\gamma}{W} \to \underset{\gamma'}{W}$$

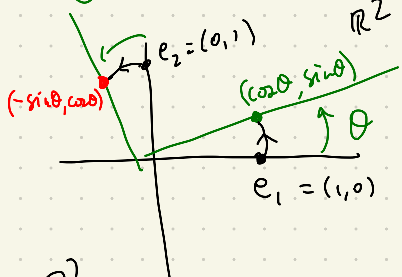
$$= {}_{\gamma'}[I_W]_{\gamma} \; {}_{\gamma}[T]_{\beta} \; {}_{\beta}[I_V]_{\beta'}$$



Ex.: 2d rotation $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$

$R(e_1) = (\cos\theta) e_1 + (\sin\theta) e_2$

$R(e_2) = (-\sin\theta) e_1 + (\cos\theta) e_2$.



for a general vector $v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

$$R(v) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (\cos\theta) x - (\sin\theta) y \\ (\sin\theta) x + \cos\theta \, y \end{pmatrix}$$

R has been described using basis $(e_1, e_2) = \beta$

i.e. $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = {}_\beta[R]_\beta$

Now let's view R as a linear map $\mathbb{C}^2 \to \mathbb{C}^2$!

and change basis to $\gamma = ((1,i), (1,-i))$.

(a basis for $\mathbb{C}^2$).

$w_1 = 1\cdot(1,0) + i(0,1)$ $\qquad$ $w_2 = 1\cdot(1,0) - i\cdot(0,1)$

$${}_\gamma[R]_\gamma = {}_\gamma[I \; R \; I]_\gamma$$

$$= {}_\gamma[I]_\beta \, {}_\beta[R]_\beta \, {}_\beta[I]_\gamma$$

$((1,0),(0,1))$ $\qquad$ $((1,i),(1,-i))$

$$= {}_\gamma[I]_\beta \, {}_\beta\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_\beta \, {}_\beta[I]_\gamma$$

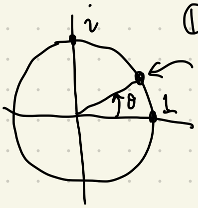
std basis

$${}_\gamma[I]_\beta = \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix} \qquad {}_\beta[I]_\gamma = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$(1,0) = \frac{1}{2}(1,i) + \frac{1}{2}(1,-i)$.

$(0,1) = -\frac{i}{2}(1,i) + \frac{i}{2}(1,-i)$.

$$\begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$



$\mathbb{C}$ $\quad e^{i\theta} = \cos\theta + i\sin\theta$

$$= \begin{bmatrix} \underbrace{\cos\theta - i\sin\theta}_{= e^{-i\theta}} & 0 \\ 0 & \underbrace{\cos\theta + i\sin\theta}_{= e^{i\theta}} \end{bmatrix} \quad \text{"diagonal matrix"}$$

$\Rightarrow$ in the basis $(1, i)$ $(1, -i)$ the
"rotation" scales $(1,i)$ by $e^{-i\theta} \in \mathbb{C}$
" $(1,-i)$ by $e^{i\theta} \in \mathbb{C}$.

eigenvector

eigenvalue

<u>Idea:</u> in standard basis, R is not diagonal, but in the new basis it is.

"Bookkeeping"

$$v_1 = e_1 + 3e_2 \qquad \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$
$$v_2 = 2e_1 - e_2$$

initial list.

$$\downarrow \text{shear}$$

$$v_1' = v_1$$
$$v_2' = v_2 - 2v_1 \qquad \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix}$$

$$\downarrow \text{scale}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\downarrow \text{shear}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

final list $\quad v_1^{final} = e_1$

$v_2^{final} = e_2$

$(v_1^{final}, v_2^{final})$ has the same span as (v_1, v_2).

Q: can we express $(v_1^{final}, v_2^{final})$ in terms of (v_1, v_2).

How to keep track of the operations used in GE?

Bookkeeping: Init

$$\left[\begin{array}{ccccc} v_1 & & = e_1 & + & 3e_2 \\ & v_2 & = 2e_1 & - & e_2 \end{array}\right]$$

1st step

$$\left[\begin{array}{ccccc} v_1 & & = e_1 & + & 3e_2 \\ -2v_1 & v_2 & = 0 \cdot e_1 & - & 7e_2. \end{array}\right]$$

Scaling

$$\begin{bmatrix} v_1 + 0 v_2 = e_1 + 3e_2 \\ \frac{2}{7} v_1 - \frac{1}{7} v_2 = 0 \cdot e_1 + 1 e_2 \end{bmatrix}$$

Efficient:

$$\underbrace{\left[\begin{array}{cc|cc} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right]}_{\text{Bookkeeping matrix.}} \xleftarrow{\text{interpret}} \begin{array}{l} v_1 = e_1 + 3e_2 \\ v_2 = 2e_1 - e_2 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 3 \\ -2 & 1 & 0 & -7 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 3 \\ 2/7 & -1/7 & 0 & 1 \end{array}\right) \quad \text{RE}$$

$$\downarrow$$

$$\left(\begin{array}{cc|cc} \frac{1}{7} & 3/7 & 1 & 0 \\ 2/7 & -1/7 & 0 & 1 \end{array}\right) \quad \text{RRE.}$$

interpret

$$\frac{1}{7} v_1 + \frac{3}{7} v_2 = e_1$$

$$\frac{2}{7} v_1 - \frac{1}{7} v_2 = e_2$$

Row operations and Bookkeeping

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ & & A & & \\ & & & & \end{bmatrix} \begin{bmatrix} \cdot & & \\ \cdot & B & \\ \cdot & & \\ \cdot & & \\ \cdot & & \end{bmatrix} = \begin{bmatrix} & \cdot & \\ & C & \\ & & \end{bmatrix}$$

$\Rightarrow$ If we do a row operation on A and then multiply, this does the same row operation on $C = AB$.

Corollary:

$$\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix} \overset{\longleftarrow n \longrightarrow}{\begin{bmatrix} \text{———} \\ \text{———} \\ \text{———} \\ \text{———} \\ \text{———} \end{bmatrix}} \begin{array}{c} \uparrow \\ k \\ \downarrow \end{array}$$

$$I \cdot B = B$$

a row op. on $B = IB$

is same as $(\text{row op. on } I) \cdot B$.

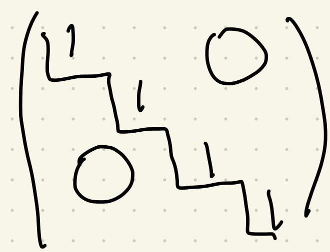
$$\Rightarrow \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{implements } R_1 \leftrightarrow R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{31} & a_{31} & a_{34} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{implements scaling.}} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{31} & a_{31} & a_{34} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{implements shear}$$

$\Rightarrow$ each e.r.o. consists of Left mult. by "elementary matrix" of the corresponding type.

$$
\begin{array}{c}
\overset{k\times n}{A} \\
\Big\downarrow \text{ first row op.} \\
\overset{k\times k}{E_1} A \\
\Big\downarrow \text{ second} \\
E_2 E_1 A \\
\vdots \\
\downarrow
\end{array}
$$



$$\overbrace{\underbrace{E_N \cdots\cdots E_1}} \underline{A} = (R)RE = \underline{I}$$

If we want to keep track of these operations. we should apply them to I as well as A.

$$\left[\ \underbrace{E_N \cdots E_1}_{\text{remembers the operation used.}}\, I \;\middle|\; E_N \cdots E_1 A\ \right]$$

$$\text{Shear}(A) = \underbrace{(\text{Shear } I)}\cdot A$$

$$\underbrace{\begin{array}{c} i_2 \\ \\ \\ \\ i_1 \end{array}\begin{bmatrix} 1 & 0 & 0 & & \lambda & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \end{bmatrix}} \cdot \underline{A}$$

$$_\gamma[I]_\beta \, _\beta[I]_\gamma = \, _\gamma[I \circ I]_\gamma$$
$$= \, _\gamma[I]_\gamma$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Assignment 7</u> $\quad \mathbb{F}^{n \times n} \quad$ $n \times n$ matrices

v.sp. over $\mathbb{F}$ $\quad V \quad \dim V = n$

$\quad\quad\quad\quad\quad W \quad \dim W = k$.

$L(V, W)$ = linear maps $V \to W$
$\quad \| $
Hom(V, W)

<u>Claim:</u> this set natrually inherits the str of a v.sp. over $\mathbb{F}$.

$T_1, T_2 \in L(V, W)$
$\lambda \in \mathbb{F}$

$$\underbrace{(\lambda \cdot T_1 + T_2)(\underset{V}{v}) = \lambda \underset{W}{\cdot} T_1(v) \underset{W}{+} T_2(v)}_{\text{definition of the operations on } L(V,W).}$$

What is the dimension of $L(V, W)$? Can we produce a basis?

<u>Claim:</u> If we choose a basis $\beta = (v_1, \dots, v_n)$ for V
and a basis $\gamma = (w_1, \dots, w_k)$ for W $\Big\} \Rightarrow$ basis for $L(V, W)$.

$$T(v_j) = \sum_{i=1}^{k} a_{ij} w_i \qquad (a_{ij}) \quad k \times n \text{ matrix}$$

$$\uparrow$$

$\mathbb{F}^{k \times n}$ $k \times n$ matrices.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix} \leftrightarrow (a_{11}, a_{12}, \ldots, \ldots, a_{kn}) \in \mathbb{F}^{kn}$$

E_{ij} (i,j fixed) $: V \to W$

$$v_p \longmapsto \begin{cases} w_i & \text{when } p = j \\ 0 & \text{else} \end{cases}$$

$$_\gamma[E_{ij}]_\beta = \begin{array}{c} \\ \\ i \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} & j & \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

(E_{ij}) is a sequence of linear maps, claim it is a basis $L(V,W)$.

$$\Big(\sum_{\substack{p=1,\ldots,k \\ q=1,\ldots,n}} a_{pq} E_{pq}\Big)(v_j) = \sum_{p=1}^{k} a_{pq} w_p = T(v_j)$$

zero unless $q=j$

w_p when $q=j$

this shows (E_{pq}) span $L(V,W)$.

check that (E_{pq}) is linearly indep

$$T = \sum_{\substack{p=1,\ldots,k \\ q=1,\ldots,n}} a_{pq} E_{pq}$$

$\mathbb{F}$ basis $\Leftarrow$

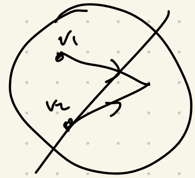
size of basis is $k \times n$. $\Rightarrow \dim L(V,W) = kn$.

Injective / Surjective / Isomorphisms $\qquad V, W$ f.d. v.s $/ \mathbb{F}$

$$\begin{array}{ccc} V & \xrightarrow{\quad T \quad} & W \\ \cup\!\mid & & \cup \\ \mathrm{Null}(T) & & \mathrm{Range}(T) \end{array}$$

linear map.

Inj.



Def: • $\mathrm{Null}(T) = \{ v \in V : Tv = 0 \} \underset{\text{lin subsp.}}{\subseteq} V$

$\overset{\shortparallel}{\mathrm{Ker}(T)}$

• $\mathrm{Range}(T) = \{ w \in W : \exists v \in V \ Tv = w \} \underset{\text{lin}}{\subseteq} W$

$\overset{\shortparallel}{\mathrm{Im}(T)}$

Prop: T injective $\Leftrightarrow \mathrm{Null}(T) = \{0\}$.

$\Leftarrow$ If $T(v_1) = T(v_2)$ then $T(v_1) - T(v_2) = 0$

(no meaning for maps of sets)

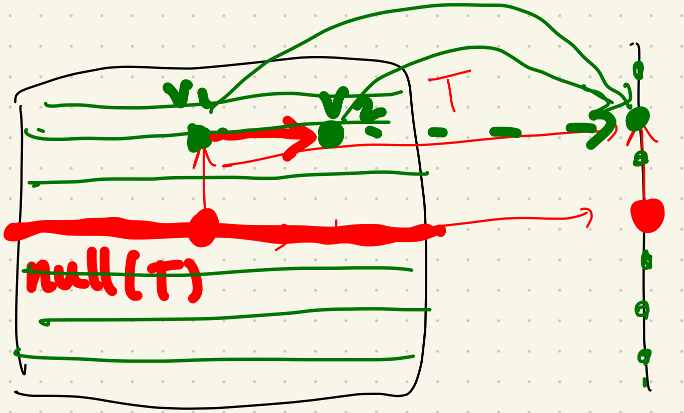
and so $T(v_1 - v_2) = 0$

i.e. $v_1 - v_2 \in \mathrm{Null}(T) \overset{\text{assumption}}{=} \{0\}$

i.e. $v_1 = v_2$

$\Rightarrow$ If T injective and $v \in \mathrm{Null}(T)$ then

$T(v) = 0 = T(0) \Rightarrow v = 0$.



$T\begin{pmatrix}1\\0\end{pmatrix} = 0$

$T\begin{pmatrix}0\\1\end{pmatrix} = 1$

$\begin{bmatrix} 0 & 1 \end{bmatrix}$

"

${}_{\beta_1}[T]_{\beta_2}$

$$\mathbb{R}^2 \xrightarrow{\quad T \quad} \mathbb{R}$$

$\text{Null}\,T = \text{Span}\begin{pmatrix}1\\0\end{pmatrix}$

Def: T is surjective when $\text{Range}(T) = W$.

"same shape"

Def: T is an isomorphism when it is inj & surjective. (linear bijection)

"isomorphism from V to W"

"V, W are isomorphic"

V, W are "equivalent"

Def: $T \in L(V,W)$ is **invertible**

when $\exists \ S \in L(W,V)$ s.t. $\begin{cases} ST = I_V \\ TS = I_W \end{cases}$

when this S exists, it is **unique** and we denote it by T^{-1}

Prop: T invertible $\Longleftrightarrow$ T injective & surjective

$\Longleftrightarrow$ $\operatorname{Null} T = 0$ and $\operatorname{Range} T = W$.

Pf: $\Rightarrow$ to show

injective: if $Tv_1 = Tv_2$

$$\Downarrow$$

$$T^{-1}(Tv_1) = T^{-1}(Tv_2)$$

$$v_1 = v_2 \quad \square \text{ inj}$$

Surjective: Let $w \in W$ then

$$T(T^{-1}w) = w \quad \square \text{ surj}$$

$\Leftarrow$ Assume inj + surj. prove invertible

Define the inverse: Let $w \in W$,

surj $\Rightarrow \exists\ v \in V \quad Tv = w$

inj $\Rightarrow$ v is unique st. $Tv = w$.

$\Rightarrow$ define $S(w) = v$.

it is an inverse:

$$\begin{cases} T(Sw) = T(v) = w \\ S(Tv) = S(w) = v. \end{cases}$$

must show S is linear.

if $Sw_1 = v_1$ want to show

$Sw_2 = v_2$ $\quad S(\lambda w_1 + w_2) = \lambda v_1 + v_2$

use linearity of T

$$\underbrace{T(\lambda v_1 + v_2)}\ = \underbrace{\lambda w_1 + w_2}.$$

$$S(\lambda w_1 + w_2) = \lambda v_1 + v_2.$$

- A vector $u \in V$ defines a linear map

$$\begin{aligned} \mathbb{F} &\xrightarrow{T_u} V \\ \lambda &\longmapsto \lambda u \end{aligned}$$

$\text{Range}(T_u) = \text{Span}(u)$.

- A list of vectors $(u_1, \dots, u_m)$ defines

$$\begin{aligned} \mathbb{F}^m &\xrightarrow{T_{(u_1, \dots, u_m)}} V \\ (a_1, \dots, a_m) &\longmapsto a_1 u_1 + \dots + a_m u_m \end{aligned}$$

$$\text{Range}(T_{(u_1, \dots, u_m)}) = \text{Span}(u_1, \dots, u_m).$$

- If the list $(u_1, \dots, u_m)$ is linearly dependent, what does this say about $T_{(u_1, \dots, u_m)}$?

that it has $\mathrm{Null}\, T \neq \{0\}$.

$\mathrm{Null}(T_{(u_1, \dots, u_m)}) = \{0\}$

iff $(u_1, \dots, u_m)$ is lin. independent.

- the list spans V

$\Longleftrightarrow$ the map $T_{(u_1, \dots, u_m)}$ has Range $= V$

is surj.

- $T_{(u_1, \dots, u_m)}$ is invertible (isomorphism) iff $(u_1, \dots, u_m)$ is a basis.

$\Rightarrow$ The choice of a basis of V is the choice of isomorphism $\mathbb{F}^m \to V$