Sets & Maps

Alg. structures (Groups, Fields ...)

Vector spaces / $\mathbb{F}$

- Linear (In) dependence and span.
- Dimension, bases
- Subspaces & operations on them
- Gauss Elimination algorithm
- Quotients V/U

Top 3 reco's for Reading Week

1. SAD Lamp.
2. Exercise outside
3. Sleep

Linear maps $L(V,W)$

- Null, Range
- Isomorphism (Invertible maps).
- Matrices, change of basis.
- Finding inverses by Bookkeeping. EROs.
- Solving Linear (in) homogeneous systems.

$$_{\gamma'}[T]_{\beta'} = {}_{\gamma'}[I]_{\gamma}\ {}_{\gamma}[T]_{\beta}\ {}_{\beta}[I]_{\beta'}$$

- Linear maps $L(V,V)$ (classify / understand);
  - ~~fixed vectors~~ eigenvectors + eigenvalues + generalized eigenspaces ...

<u>Classification</u>: Jordan canonical form.

- Determinants.

Duality

Def: Let V v.sp. / $\mathbb{F}$. The dual space, V^*, is

$$\boxed{V^* = L(V, \mathbb{F}).}$$

i.e. the space of linear functions
"functionals"
"linear forms"
"covectors"

$\alpha_1, \alpha_2 \in V^*$ — represented by matrices $\mathbb{F}^{k \times n}$, $n = \dim V$, $k = \dim W$

as for any $L(V, W)$, V^* is a v.space.

$$(\lambda \cdot_{V^*} \alpha_1 +_{V^*} \alpha_2)(\underset{\underset{V}{\cap}}{v}) = \lambda \cdot_{\mathbb{F}} \alpha_1(v) +_{\mathbb{F}} \alpha_2(v)$$

Prop: Suppose V is finite dimensional, then $\dim V^* = \dim V$.

Pf: construct a basis for V^* by first choosing a basis for V

$\beta = (e_1, \ldots, e_n)$ basis for V.

If $\alpha \in V^*$ $\qquad \alpha(v) = ? \qquad v = \sum_{i=1}^n a_i e_i$

$$\alpha(v) = \alpha\left(\sum_{i=1}^n a_i e_i\right)$$

$$= \sum_{i=1}^n a_i \, \alpha(e_i)$$

If we know value of α on e_i we can determ. value on V.

α is determined by n numbers in $\mathbb{F}$ $(\alpha(e_1), \ldots, \alpha(e_n))$

build a basis for V^* as follows: e_1^* defined by
$$\begin{cases} e_1 \longmapsto 1 \\ e_2 \longmapsto 0 \\ \vdots \\ e_n \longmapsto 0 \end{cases}$$

$$e_2^* : \begin{cases} e_1 \mapsto 0 \\ e_2 \mapsto 1 \\ e_3 \mapsto 0 \\ \vdots \\ e_n \mapsto 0. \end{cases}$$ and so on $\quad e_j^*(e_i) = \begin{cases} 1 & i=j \\ 0 & \text{else}. \end{cases}$

Claim: $\beta^* = (e_1^*, \ldots, e_n^*)$ is a basis for V^*

Spans: Let $\alpha \in V^*$ define $a_i = \alpha(e_i) \quad \forall i.$

then. $\alpha = (a_1)e_1^* + (a_2)e_2^* + (a_3)e_3^* + \cdots + (a_n)e_n^*$

$\Rightarrow \beta^*$ spans V^*

Lin. Indep: $\sum_{i=1}^{n} b_i e_i^* = 0$ Want to show $b_i = 0 \ \forall i.$

but $\underbrace{\left(\sum_{i=1}^{n} b_i e_i^*\right)}_{0}(e_j) = b_j \Rightarrow b_j = 0 \ \forall j.$

$$\left(\sum_{i=1}^{n} b_i e_i^*\right) = 0 \qquad 0$$

$\square$

We can even use this to define an isomorphism.

$$V \xrightarrow{T} V^*$$

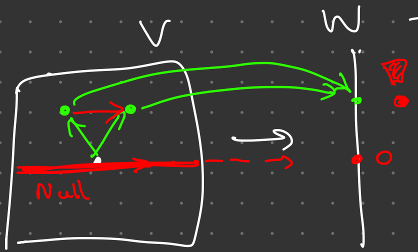
by defining $T(e_i) = e_i^* \quad \forall i$

has zero null space $\Rightarrow$ injective $\quad \dim V = V^* \Rightarrow T$ invertible !!

Warning: T depends on the choice of β !! No natural isomorphism

$T : V \longrightarrow W$ linear.

Suppose we have $x, y \in V$ distinct s.t. $T(x) = T(y)$



$$\Downarrow$$

$$T(x) - T(y) = 0$$

$$\Downarrow \text{ Lin. of } T$$

$$T(\underbrace{x-y}) = 0.$$

$$\Downarrow$$

nonzero elt. $\quad x - y \in \mathrm{Null}(T)$

$$\Downarrow$$

$$\mathrm{Null}(T) \neq \{0\}.$$

Duality applies not only to v.spaces $V \rightsquigarrow V^*$ but also to linear maps.

Let $A : V \longrightarrow W$ linear map.

Let V^*, W^* be dual spaces.

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[dr, "f \circ A"'] & W \arrow[d, "f"] \\ & \mathbb{F} \end{tikzcd}$$

A^*

If $f \in W^*$

Then $f \circ A \in V^*$

$$\shortparallel$$

$$(A^*)(f)$$

by defn of A^*

$$\overset{n}{V} \xrightarrow{A} \overset{k}{W} \qquad \text{Linear map}$$

$$\overset{n}{V^*} \xleftarrow{A^*} \overset{k}{W^*} \qquad \text{Dual linear map.}$$

"Contra variance"

Suppose $\beta = (e_i)$ basis for V $\Rightarrow$ $\beta^* = (e_i^*)$ V^*

$\gamma = (f_i)$ " " W. $\Rightarrow$ $\gamma^* = (f_i^*)$ W^*

$$_{\gamma}[A]_{\beta} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix} = M$$

Claim:

$$_{\beta^*}[A^*]_{\gamma^*} = \begin{bmatrix} a_{11} & \cdots & a_{k1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{kn} \end{bmatrix} = M^T$$

The matrix of A^* is the transpose of the matrix of A.

$$\mathrm{Tr} : \mathbb{F}^{n \times n} \longrightarrow \mathbb{F}$$

$$\mathrm{Tr} \in (\mathbb{F}^{n \times n})^*$$

$$\mathrm{Tr}(M) \in \mathbb{F}$$

$$\underbrace{\mathbb{F}^n}_{\beta = (e_1, \dots, e_n)} \xrightarrow{M} \mathbb{F}^n \quad \beta = (e_1, \dots, e_n)$$

$$M = {}_\beta[M]_\beta$$

γ diff. basis for $\mathbb{F}^n$

$${}_\gamma[M]_\gamma = {}_\gamma[I]_\beta \, {}_\beta[M]_\beta \, {}_\beta[I]_\gamma$$

$$= P M P^{-1}$$

changing basis simultaneous on dom + codom.

P has columns = coeffs of β in terms of γ

$$\operatorname{Tr} M = \operatorname{Tr}(PMP^{-1})$$

$v \in V$ $\qquad v = \sum a_i e_i$

representing v as a list $(a_1, a_2, \ldots$ of numbers.

choose

$\alpha \in V^*$ a linear functional,

$\alpha(v)$ single number! $\in \mathbb{F}$.

$(\alpha_1, \ldots, \alpha_n)$ basis for V^*

$$\left(\alpha_1(v), \ldots, \alpha_n(v)\right)$$

Thm: If V is f.d. v.sp. then $(V^*)^*$ is naturally isomorphic to V, i.e., we can define an isomorphism $V \to (V^*)^*$ which does not depend on any arbitrary choices (of a basis e.g.).

$$V \xrightarrow{D} (V^*)^*$$

$v \in V$ $\quad D(v) = ?$ ← a linear f^n on (linear f^{ns} on V)

i.e. given $\alpha \in V^*$ (a linear f^n on V).

$$(D(v))(\alpha) \in \mathbb{F}$$

$$(Dv)(\alpha) = \alpha(v) \in \mathbb{F}$$

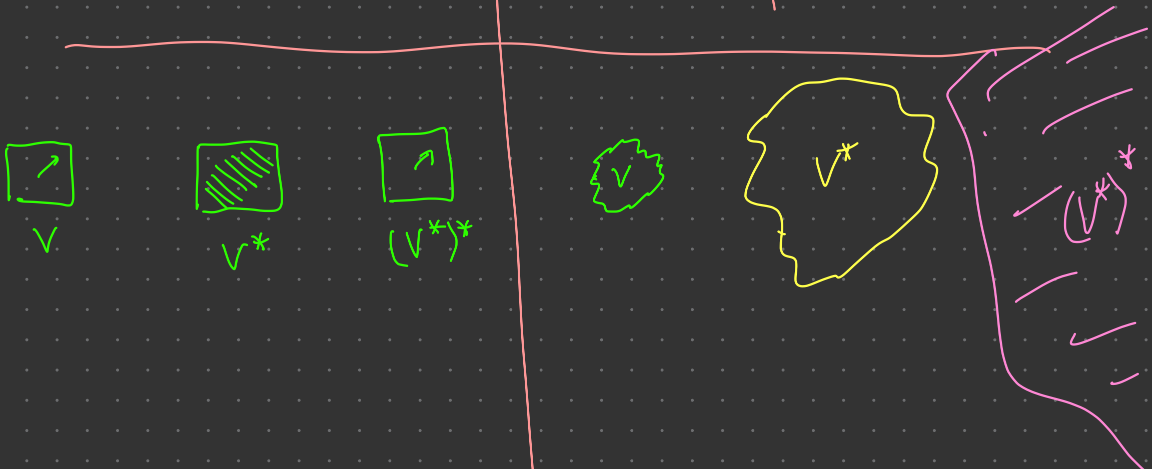
(Dv) is the linear f^n which takes α to its evaluation at $v \in V$ (v fixed).

$$Dv = Ev_v = \text{evaluation at } v \in V$$

Thm: the map $v \longmapsto Dv$ is an isomorphism!!

Warning: If V is not finite dimensional, then V is never isomorphic to V^*, or to $(V^*)^*$.

Finite dim'l	Infinite Dim'l



$\alpha \in V^* = \boxed{L(V, \mathbb{F})}$ $\dim V = n$

β basis for V

$$[\alpha]_\beta = [\alpha(e_1) \quad \alpha(e_2) \quad \cdots \quad \alpha(e_n)]$$

$1 \times n$ matrix "row vector"

$$[a_1, \cdots, a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = a_1 x_1 + \cdots + a_n x_n$$

$\in \mathbb{F}$

represents a dual vector

$a_1 e_1^* + \cdots + a_n e_n^*$

represents a vector in V

$x_1 e_1 + \cdots + x_n e_n$

$$L(L(V, \mathbb{F}), \mathbb{F}) = (V^*)^* = V$$

If V is finite dim^l: $V \underset{\text{duality}}{\overset{\text{duality}}{\rightleftarrows}} V^*$

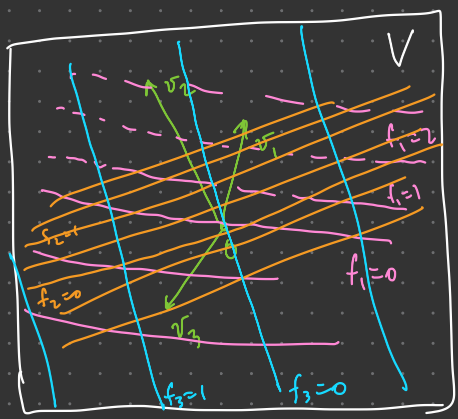
any vector $v \in V$ may be viewed as

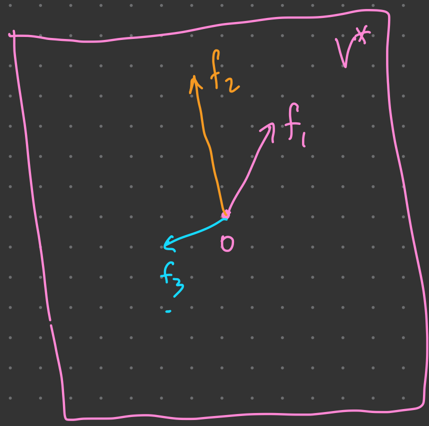
a linear map $\mathbb{F} \longrightarrow V$

$$\lambda \longmapsto \lambda \cdot v$$

$$1 \longmapsto v$$

$$V \cong L(\mathbb{F}, V) \qquad L(V, \mathbb{F}) = V^*$$





$$\boxed{f_1 = a_1 x_1 + \cdots + a_n x_n}$$

what is an elt of $(V^*)^*$?

it takes linear f^{ns} (on V)

to $\mathbb{F}$

claim: each $v \in V$ defines

a special element in here

i.e. it takes the

linear f^n $(f : V \to \mathbb{F})$

to the number

$f(v) \in \mathbb{F}$.