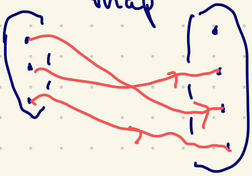
Finite Sets	Finite-dim v.spaces / $\mathbb{F}$
X $\rightsquigarrow$ construct	$\mathbb{F}(X)$ = formal linear combos of elts of X.
$X \xrightarrow[\text{map}]{f} Y$ $\rightsquigarrow$ construct	$\rho_f : \mathbb{F}(X) \longrightarrow \mathbb{F}(Y)$. Linear map.



$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & Z \end{tikzcd}
\qquad \rightsquigarrow \qquad
\begin{tikzcd} & \mathbb{F}(Y) \arrow[dr, "\rho_g"] & \\ \mathbb{F}(X) \arrow[ur, "\rho_f"] \arrow[rr, "\rho_g \circ \rho_f"'] & & \mathbb{F}(Z) \end{tikzcd}$$

$$\rho_g \circ \rho_f = \rho_{g \circ f}$$

"compos. of linear maps of f, g is the linear map of compos. of f, g"

Functor

from Sets to Vector Spaces

↖ categories ↗

Review: V, W f.d. v.sp/$\mathbb{F}$ $\Rightarrow$ $L(V,W) \ni A$ can be described as a $\dim W \times \dim V$ matrix of elts in $\mathbb{F}$: choose bases β for V, γ for W and write

($\dim W = k$, $\dim V = n$)

$$M(A, \beta, \gamma) = {}_{\gamma}[A]_{\beta} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix}$$

Changing β, γ will affect the matrix to β', γ' $\Rightarrow$

$$ {}_{\gamma'}[A]_{\beta'} = \underbrace{\overset{k\times k}{{}_{\gamma'}[I_W]_{\gamma}}}_{\substack{k\times k \text{ invertible matrix} \\ P}} \; \overset{k\times n}{{}_{\gamma}[A]_{\beta}} \; \underbrace{\overset{n\times n}{{}_{\beta}[I_V]_{\beta'}}}_{\substack{n\times n \text{ invertible matrix} \\ Q^{-1}}} \qquad Q = {}_{\beta'}[I]_{\beta}$$

To emphasize the fact that the initial and final $k \times n$ matrices represent the same linear map, we can define two matrices A, A' to be "equivalent" when $\exists$ P, Q invertible matrices s.t.

$$A' = PAQ^{-1}$$

Implements el. column operations.

$$\overset{\text{invertible } k\times k}{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \lambda & \\ & & & 1 \end{bmatrix}} \overset{k\times n}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}} \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \lambda & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

implements ERO on A.

Simplify given $k \times n$ matrix A without changing its equivalence class.

① GE (Row ops) to RRE

$$\begin{bmatrix} 1 & * & 0 & 0 & * & * & 0 & * \\ & & 1 & 0 & * & * & 0 & * \\ & & & 1 & * & * & 0 & * \\ & & & & & & 1 & * \\ & 0 & & & & & & \end{bmatrix}$$

② GE (Col. ops) to clear all unknowns (*)

result

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 & 0 \\ & & & & & & 1 & 0 \\ & 0 & & & & & & \end{bmatrix}$$

③ Column switching to bring everything to right.

$$\begin{bmatrix} & & \boxed{1} & & & 0 \\ & & & \boxed{1} & & \\ & 0 & & & \boxed{1} & \\ & & & & & \boxed{1} \end{bmatrix}$$

offset diagonal

Thm:

Every $k \times n$ matrix A is equiv. to block matrix

$$\left[\begin{array}{c|c} 0 & I \\ \hline 0 & 0 \end{array}\right]$$

I: $r \times r$

$r = \operatorname{rk} A$

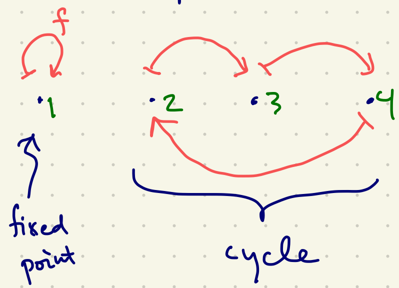
In terms of bases: for any linear map $A : V \to W$

$\exists$ bases $\beta = (e_1, \dots, e_n)$ of V, $\gamma = (f_1, \dots, f_k)$ of W s.t.

$$A(e_1) = f_1$$
$$A(e_2) = f_2$$
$$\vdots \qquad \vdots$$
$$A(e_r) = f_r$$
$$A(e_i) = 0 \quad \forall i > r$$

$$_\gamma[A]_\beta = \left[\begin{array}{c|c} I_{r\times r} & 0 \\ \hline 0 & 0 \end{array}\right]$$

Recall studied maps
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & X \arrow[r, "g"] & X \end{tikzcd}$$
self-maps of a set.



represent the map f. in cycle notation (234)

There is a corresponding description for Linear operators

"Linear operators on V" $= L(V,V)$.

<u>Key difference:</u> since domain + codomain coincide, we need not choose two bases; one basis for V suffices.

i.e. $A \in L(V,V)$ has a matrix after single choice $\beta = (e_1, \dots, e_n)$ basis for V.

$$_\beta[A]_\beta = \begin{bmatrix} & & \\ & \begin{array}{c} n\times n \\ \text{(square)} \\ \text{matrix.} \end{array} & \\ & & \end{bmatrix}$$

$\Rightarrow$ have a different equivalence relation on $n\times n$ matrices

Def: Two $n \times n$ matrices A, A' are Similar or Conjugate when $\exists$ P invertible $n \times n$ s.t.

$$\boxed{A' = PAP^{-1}}$$

Motivation: If we change β to β' then

$${}_{\beta'}[A]_{\beta'} = \underbrace{{}_{\beta'}[I]_{\beta}}_{P} \; {}_{\beta}[A]_{\beta} \; \underbrace{{}_{\beta}[I]_{\beta'}}_{P^{-1}}$$

Inverse!

[The fact that column ops are forced by the row ops means that simplification does not occur by G.E.]

$\Rightarrow$ new approach is needed to "understand" the operator: Spectral theory

eigenvector analysis

What kind of operators are there?

$L(V,V)$.

0) $0 : v \longmapsto 0 \ \forall v.$ $\qquad {}_\beta[0]_\beta = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$ indep. of basis!

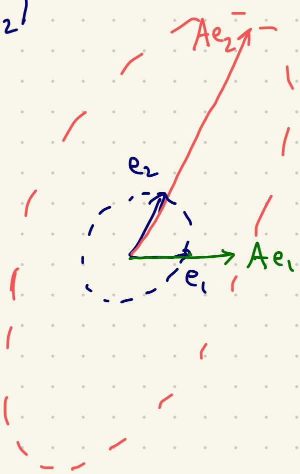
1) $I : v \longmapsto v \ \forall v$ $\qquad {}_\beta[I]_\beta = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}$

2) $\lambda \in \mathbb{F}$ $\quad \lambda I : v \longmapsto \lambda v$ $\qquad {}_\beta[\lambda I]_\beta = \begin{bmatrix} \lambda & & & 0 \\ & \lambda & & \\ & & \lambda & \\ 0 & & & \lambda \end{bmatrix}$

3) $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ ← operator on $\mathbb{F}^2$ written in std basis

$\parallel$

${}_\beta[A]_\beta$



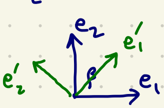
$e_1 = (1,0)$
$e_2 = (0,1)$

$\boxed{\beta = (e_1, e_2)}$

While this operator is simple / easy to understand when written as above, if we change basis, won't be simple:

eg $\left[\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \end{array}\right.$

$$\underbrace{\begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}}_{P^{-1}} = \underbrace{\begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}}_{\text{same operator! different basis!}}$$



$${}_{\beta'}[I]_\beta \ {}_\beta[A]_\beta \ {}_\beta[I]_{\beta'}$$

$$\beta' = \left(\begin{pmatrix} \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt2 \\ 1/\sqrt2 \end{pmatrix} \right) = (e_1', e_2').$$

In the above example,

e_1, e_2 play the role of "fixed pts"

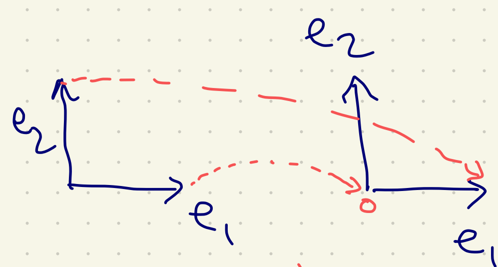
e_1', e_2' <u>not</u>

e_1, e_2 are examples of Eigenvectors.

the scale factors λ_1, λ_2 are examples of Eigenvalues

4) besides eigenvectors, there is another type of behaviour : <u>cycles</u>

<u>e.g.</u> $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$



one eigenvect.

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$e_1 \longmapsto 0$

$e_2 \longmapsto e_1$

$e_3 \longmapsto e_2$

$\beta = (e_1, \dots, e_n)$ basis for V

$\beta' = (e_1', \dots, e_n')$ $\quad e_i' \in V^*$

$e_1' \, (e_1) = 1$

$(e_2) = 0$

$\vdots$

$(e_n) = 0$

$\boxed{e_i'(e_j) = 1 \;\; i=j \\ 0 \text{ else}}$

Prop

β' is lin. indep: let

$a_1 e_1' + \dots + a_n e_n' = 0$

$= 0$

WTS $a_i = 0 \;\; \forall i$.

strategy: Show $a_1 = 0$, then show $a_2 = 0$, ...

apply LHS to e_1: $\quad a_1 e_1'(e_1) + a_2 \cancel{e_2'(e_1)} + \cdots$

1

$= a_1$

RHS to e_1: $\quad 0(e_1) = 0.$ $\Big\} \Rightarrow \boxed{a_1 = 0}$

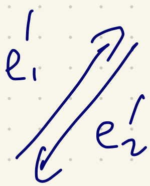
apply LHS to e_2 $\quad a_1 \cancelto{0}{e_1'(e_2)} + a_2 \overset{1.}{e_2'(e_2)} + \cancelto{0}{\cdots}$

$$= a_2$$

RHS to e_2 $= 0$,

$$\left.\begin{matrix} = a_2 \\ = 0 \end{matrix}\right\} \; a_2 = 0$$

repeat. $\quad \ldots \quad$ □



$$e_1' + e_2' = 0$$

apply LHS to e_1

$$e_1'(e_1) + e_2'(e_1) = 1 + 0$$
$$= 1.$$

RHS $= 0$

$1 = 0$ #.

Polynomials :

$$x^2 + 1 \neq (x-\lambda_1)(x-\lambda_2)$$

extend numbers

$$x^2+1 = (x-i)(x+i)$$

λ_1, λ_2 real

once the extension $\mathbb{R} \subset \mathbb{C}$ is done, we can factor all polynomials

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

$$a_i \in \mathbb{C}$$

$$= a_n (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)$$

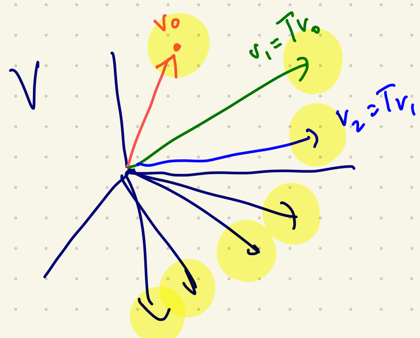
$\mathbb{C}$

V f.d.v.sp / $\mathbb{F}$

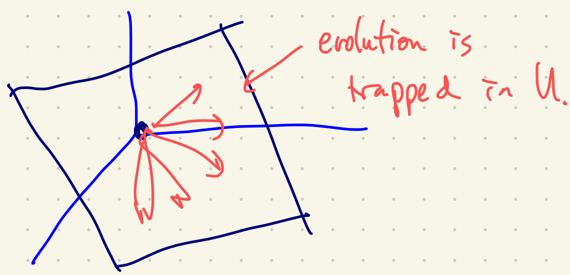
$$T \in L(V,V)$$

Def: Invariant subspace $U \subseteq V$ linear subspace

st. $T(U) \subseteq U$ i.e. $Tu \in U \quad \forall u \in U.$

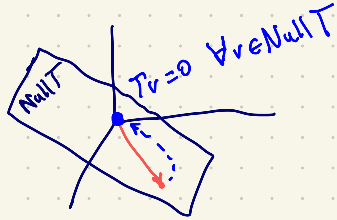


given $T \in L(V,V)$



Ex.: (0) $\{0\} \subseteq V$ invt.

(1) Null T



(2) Range(T) i.e. $\exists\, v \in V$ $u = Tv$

$\Downarrow$

u

$Tu = Tu$

$Tu \in \text{Range}(T)$.

Def^n: an eigenvalue of T is a number $\lambda \in \mathbb{F}$

s.t. $\exists\ v$ nonzero with

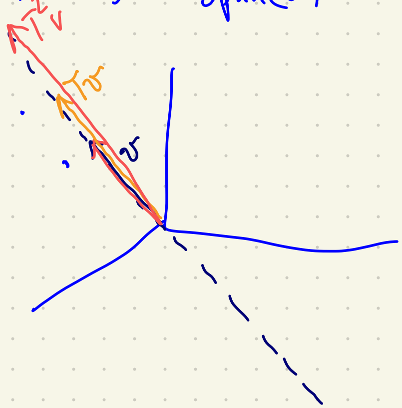
$$\boxed{Tv = \lambda v}$$

$(T - \lambda I)v = 0$

$v \in \text{Null}(T - \lambda I)$

The set of eigenvalues in $\mathbb{F}$ is called the SPECTRUM of T

notice this means Span(v) is an invariant subspace



Warning:

$0 \in \mathbb{F}$ is a valid eigenval.

Any $v \neq 0$ in $\mathrm{Null}(T)$ is an eigenvector with eigenvalue 0.

How to find eigenvectors?

Solve $\quad Av = \lambda v$

i.e. $\quad (A - \lambda I)v = 0$

2-step procedure

1. find value λ s.t. $A - \lambda I$ has nullity > 0

   i.e. $A - \lambda I$ not injective

   i.e. $A - \lambda I$ not surjective.

   i.e. $\mathrm{rank}(A - \lambda I) < \dim V$.

2. once found specific λ, "λ_1" must find nonzero

$$v \in \mathrm{Null}(A - \lambda_1 I)$$

Eg.: $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$ (note Null $A = \{0\}$ so 0 not eigenvalue.)

(1) $A - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{pmatrix}$

Q.: when does this have nonzero null space.

A: do GE

$$\downarrow$$

$$\begin{matrix} -2\lambda & 2 & 0 \\ 0 & -\lambda & 1 \\ \cancel{2\lambda} & \cancel{-5\lambda} & \cancel{\lambda(4-\lambda)} \\ \cancel{0} & \cancel{2-5\lambda} & \cancel{\lambda(4-\lambda)} \\ \cancel{0} & \cancel{2} & \cancel{\lambda(4-\lambda)-5} \\ \cancel{0} & \cancel{2\lambda} & \cancel{\lambda(\lambda(4-\lambda)-5)} \\ 0 & 0 & \lambda(\lambda(4-\lambda)-5)+2 \end{matrix}$$

factor this

$$\begin{pmatrix} -2\lambda & 2 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -(\lambda-1)^2(\lambda-2) \end{pmatrix} \xrightarrow{\lambda=1} \underbrace{\begin{pmatrix} -2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$$

vector in null space of this.

eigenvector for $\lambda = 1$

this will have nonzero Null space $\Leftrightarrow \lambda = 1$ or $\lambda = 2$.

$\Rightarrow \lambda = 1, \lambda = 2$ are the 2 eigenvalues for this operator.

(2) choose $\lambda = 1$ first plug in explicit value.

Thm: (Existence of an eigenvalue over $\mathbb{F} = \mathbb{C}$)

Pf: suppose $\dim V = n$ choose any nonzero vector $v \in V$.

$$(v, Tv, T^2 v, T^3 v, \ldots, T^n v) \leftarrow \text{MUST BE LIN DEP!!!}$$

i.e. $\exists$ a_i not all zero s.t.

$$a_0 v + a_1 T v + \cdots + a_n T^n v = 0$$

$$\underbrace{(a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n)}_{p(z)} v = 0$$

View this as a polynomial! with coeffs in field $\mathbb{C}$!

$$= a_n (T - r_1 I)(T - r_2 I) \cdots (T - r_n I) v$$

Roots of $p(z)$ in $\mathbb{C}$.

product of $(T - r_k I)$ has nonzero null space i.e. not invertible

$\Rightarrow$ at least one of the factors is not invertible, i.e. $(T - r_i I)$ has a nonzero null space

$\Rightarrow$ eigenval $\lambda_1 = r_i$.

eigenvector is ?? find it.

---

if we can find a basis of eigenvectors

$\beta = (e_1, \dots, e_n)$

Then

$$ {}_{\beta}[T]_{\beta} = \begin{bmatrix} \lambda_1 & 0 & 0 & & 0 \\ 0 & \lambda_2 & 0 & & \\ \vdots & 0 & \lambda_3 & & \\ & \vdots & 0 & \ddots & \\ 0 & 0 & 0 & & \lambda_n \end{bmatrix} $$

Diagonal Matrix.

this finding of a basis of eigenvectors is called "Diagonalization"!.

Thm

v_1, v_2 eigenvectors with eigenvalues $\lambda_1 \quad \lambda_2$ and $\lambda_1 \neq \lambda_2$.

Then (v_1, v_2) lin. indep.

Pf: suppose $a_1 v_1 + a_2 v_2 = 0$ ①

then apply T $\quad \underbrace{a_1 T v_1 + a_2 T v_2}_{a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2} = T0 = 0$

$a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 = 0$ ②

Want to show $a_1 = 0$ and $a_2 = 0$

to show $a_1 = 0$: $\quad$ take λ_2 ① − ②

$$a_1 \underbrace{(\lambda_2 - \lambda_1)}_{\neq 0} \underbrace{v_1}_{\neq 0} = 0 \Rightarrow a_1 = 0$$

$a_2 = 0$: $\quad$ take λ_1 ① − ②

$$a_2 (\lambda_1 - \lambda_2) v_2 = 0 \Rightarrow a_2 = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A$$

$$Ae_1 = e_1$$

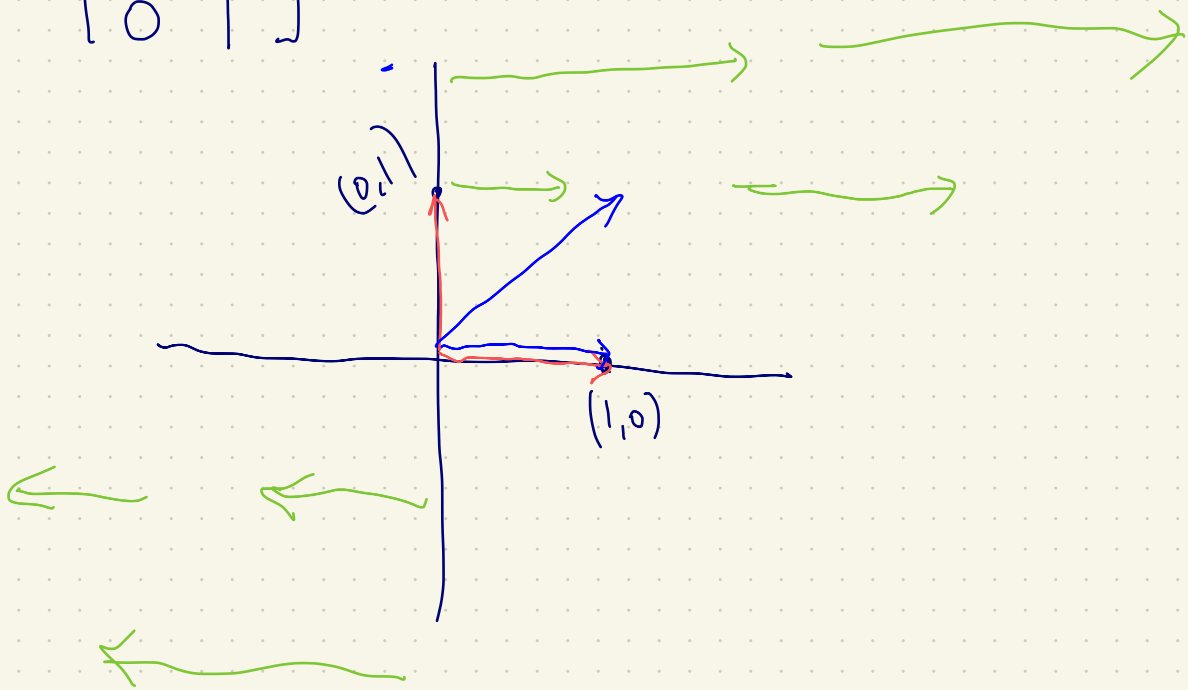
$$Ae_2 = e_2$$

single eigenvalue

2d space of eigenvectors.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$Ae_1 = e_1$$



$$\begin{bmatrix} 1 & 1 & & 0 \\ & 1 & 1 & \\ & & 1 & 1 \\ 0 & & & 1 \end{bmatrix}$$

$e_1 \longmapsto e_1$

$e_2 \longmapsto e_2 + e_1$

$e_3 \longmapsto e_3 + e_2$

$\vdots$

$e_n \longmapsto e_n + e_{n-1}$

$$A - 1I = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

$e_1 \longmapsto 0$

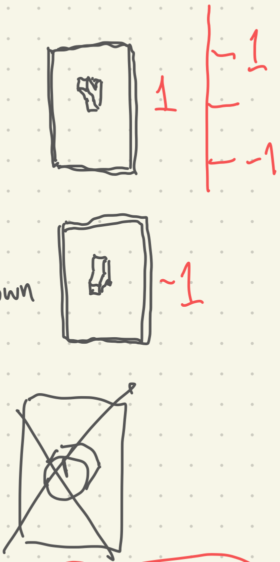
$e_2 \longmapsto e_1$

$e_3 \longmapsto e_2$

$e_4 \to e_3$

up

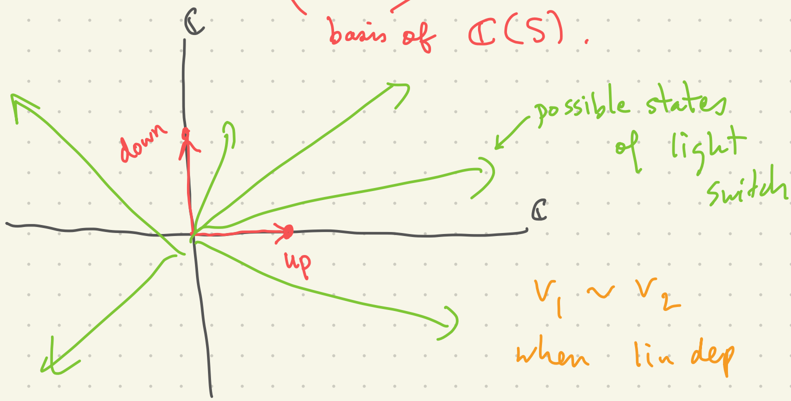
down



Space of states $= \{ \text{up}, \text{down} \} = S$

$\mathbb{C}(S) = \{ a_1 \cdot (\text{up}) + a_2 (\text{down}) \; : \; a_1, a_2 \in \mathbb{C} \}$

basis of $\mathbb{C}(S)$.



$v_1 \sim v_2$ when lin dep

$q : S \to \mathbb{C}$

$q(\text{up}) = 1$

$q(\text{down}) = -1$

In Quantum mechanics, the light switch has Quantum state $=$ a vector In $\mathbb{C}(S)$

Operator: "Position" operator Q

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

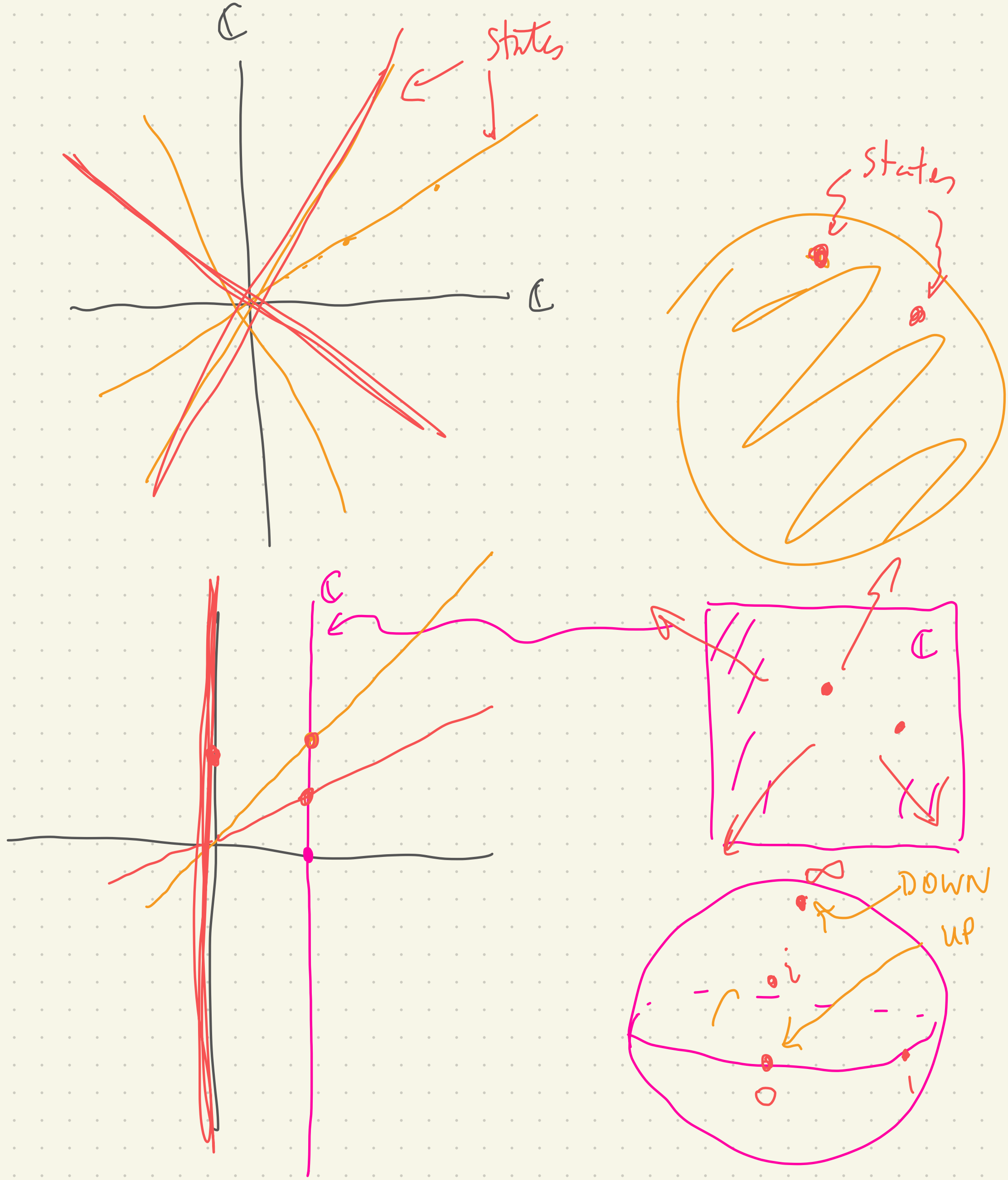
$$Q(\text{up}) = Q \begin{array}{c} \text{up} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

up is eigenvector w/ eigenval 1.

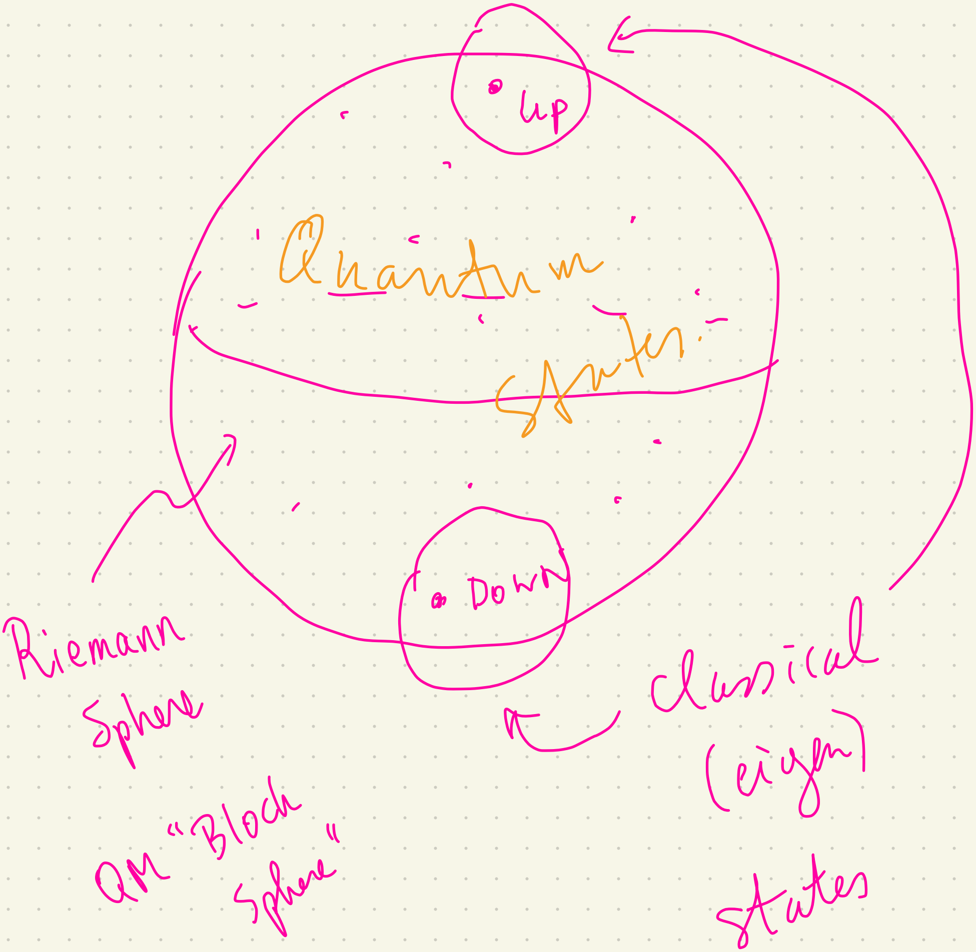
down " " " " -1.

$$Q \begin{pmatrix} -1 \\ 0 \end{pmatrix} = (1) \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

classical state up



ℂ
states
ℂ
states
ℂ
ℂ
∞
DOWN
UP
i
0



up
Quantum States
Down
Riemann Sphere
QM "Bloch Sphere"
classical (eigen) states