# Notes on the Jordan canonical form

## Introduction

Let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$, and let $T : V \longrightarrow V$ be a linear operator such that

$$(T - a_1)^{k_1} \cdots (T - a_m)^{k_m} = 0, \tag{1}$$

for $a_1, \ldots, a_m$ distinct numbers in $\mathbb{F}$. The purpose of this note is to explain how to find a Jordan basis, i.e. a basis $\beta$ for $V$ such that $[T]_\beta^\beta$ is block diagonal, where each block looks like

$$\begin{bmatrix} a & 1 & & & \\ & a & 1 & & \\ & & \ddots & \ddots & \\ & & & a & 1 \\ & & & & a \end{bmatrix} \tag{2}$$

The matrix above is called a Jordan block with eigenvalue $a$. Note that a basis $(v_1, \ldots, v_l)$ giving rise to a Jordan block is "cyclic" in a certain sense: from the shape of the matrix, we see that if we apply $T - a$ to $v_i$, we obtain $v_{i-1}$. We call $v_l$ a "cyclic" vector, since the rest of the basis can be generated from $v_l$, via $v_i = (T - a)^{l-i} v_l$.

The first thing we must do to simplify our problem is to apply the decomposition theorem, which tells us that the null space of $\prod_{i=1}^m (T - a_i)^{k_i}$, which is all of $V$ by (1), is a direct sum of the generalized[1] eigenspaces $\mathrm{null}(T - a_i)^{k_i}$. In other words, we have a direct sum decomposition

$$V = V(a_1) \oplus \cdots \oplus V(a_m), \tag{3}$$

where $V(a_i) = \mathrm{null}(T - a_i)^{k_i}$. **So, we can focus on finding a Jordan basis for each of the $V(a_i)$ separately. Then by putting the bases together we obtain a Jordan basis for $V$.** For this reason, we may assume from now on that $T$ is a linear operator on a vector space $V(a)$ and that $(T - a)^k = 0$, and we are trying to find a basis in which $T$ has diagonal blocks of the form (2).

[1] Recall that $v$ is a generalized eigenvector with eigenvalue $a$ when there exists some $i$ with $(T - a)^i v = 0$.

## What's the idea?

To better illustrate the kind of problem we are trying to solve, here are two linear operators, each with an 8-dimensional generalized eigenspace $V(a)$:

$$T_1 = \begin{bmatrix} a & 1 & & & & & & \\ & a & 1 & & & & & \\ & & a & & & & & \\ & & & a & 1 & & & \\ & & & & a & & & \\ & & & & & a & 1 & \\ & & & & & & a & \\ & & & & & & & a \end{bmatrix} \qquad T_2 = \begin{bmatrix} a & 1 & & & & & & \\ & a & 1 & & & & & \\ & & a & 1 & & & & \\ & & & a & & & & \\ & & & & a & 1 & & \\ & & & & & a & & \\ & & & & & & a & \\ & & & & & & & a \end{bmatrix}$$

So, $T_1$ has Jordan blocks of size $(3, 2, 2, 1)$, while $T_2$ has blocks of size $(4, 2, 1, 1)$. They happen to have the same number of blocks, but the size of the blocks is different. In general, we could have blocks with sizes making up any partition of 8, for example $8 = 3 + 2 + 1 + 1 + 1$.

Given $T$, we would like to figure out how many Jordan blocks there are, what their sizes are, and how to find the actual basis giving this Jordan form. The idea is simple:

$$\boxed{\begin{array}{c}\textbf{The key is the sequence of null spaces } \operatorname{null}(T-a)^i, \quad i = 1, \ldots, k\textbf{.} \\ \textbf{It is convenient to define } t_i = \dim \operatorname{null}(T-a)^i.\end{array}}$$

To see how this works, let's assume that $T$ has a Jordan form, but we don't know what it is.

- The dimension of the eigenspace $\operatorname{null}(T-a)$ tells you exactly *how many* Jordan blocks there are, since each Jordan block has a 1-dimensional eigenspace. In other words, $t_1$ is the number of Jordan blocks.

- If $T$ has only Jordan blocks of size 1, then $t_2 = \dim \operatorname{null}(T-a)^2$ is the same as $t_1$. But in general, $t_2$ will be bigger than $t_1$, because each Jordan block of size $> 1$ will contribute $+1$. Therefore $t_2 - t_1$ tells us the number of Jordan blocks of size $> 1$, and we can conclude that the number of Jordan blocks of size 1 is $t_1 - (t_2 - t_1)$.

- In general, $t_{k+1} - t_k$ is the number of Jordan blocks of size $> k$, and so the number of Jordan blocks of size exactly $k$ is $(t_k - t_{k-1}) - (t_{k+1} - t_k) = 2t_k - t_{k-1} - t_{k+1}$.

We can phrase the above discussion in a very clever way[2] as follows:

**Proposition 1.** *Let* $\mathbf{t} = (t_0, t_1, \ldots)$*, where* $t_i = \dim \operatorname{null}(T-a)^i$*. Then* $T$ *has* $s_i$ *Jordan blocks of size* $i$*, where* $\mathbf{s} = (s_0, s_1, \ldots)$ *is given by*

$$\mathbf{s} = -R(L-1)^2\mathbf{t},$$

*where* $R, L$ *are the right and left shift operators on sequences.*

For example, for $T_1$ as above, we have $\mathbf{t} = (0, 4, 7, 8, 8, \ldots)$. Hence we have $\mathbf{s} = (0, 1, 2, 1, 0, 0, \ldots)$ indicating one block of size 1, two of size 2, and one of size 3, as is indeed the case. For $T_2$ we have $\mathbf{t} = (0, 4, 6, 7, 8, 8, \ldots)$, so that $\mathbf{s} = (0, 2, 1, 0, 1, 0, 0, \ldots)$, yielding two Jordan blocks of size 1, one of size 2, and one of size 4.

[2] I learned this from Prof. John Labute, I'd like to know if appears elsewhere.

## How to find a Jordan basis

In this section we will give an algorithm for finding a Jordan basis, i.e. a basis $\beta$ for $V(a)$ such that $[T]_\beta^\beta$ is in Jordan form. Beware: there are many possible choices of Jordan basis, even though the Jordan form will always consist of the same number of Jordan blocks of each size. For example, the identity operator is in Jordan form in *any* basis.

To make our life easier, we would like to introduce the following terminology:

**Definition 1.** *Let $U \subset V$ be a linear subspace. Then we say that the list of vectors $(v_1, \ldots, v_l)$ in $V$ is "linearly independent mod $U$" when $\sum_i \alpha_i v_i \in U$ implies $\alpha_i = 0$ for all $i$. We say that $(v_1, \ldots, v_l)$ is a "basis of $V$ mod $U$" when it is linearly independent mod $U$ and $V = U + \operatorname{span}(v_1, \ldots, v_l)$.*

It is easy to find a basis of V mod U: simply choose a basis $(u_1, \ldots, u_k)$ for $U$ and extend it to a basis $(u_1, \ldots, u_k, v_1, \ldots, v_l)$ for $V$. The vectors $(v_1, \ldots, v_l)$ are then a basis of V mod U.

We now give the algorithm for finding a Jordan basis for $V(a)$. The idea is to find a cyclic vector for each Jordan block, which then generates the Jordan basis for that block. This algorithm finds the cyclic vectors for the largest Jordan blocks first, and then proceeds to find the cyclic vectors for the smaller blocks.

Step 1: Choose a basis $(v_k^1, \ldots, v_k^{s_k})$ of $V(a) = \operatorname{null}(T-a)^k$ mod $\operatorname{null}(T-a)^{k-1}$. These will be our cyclic vectors for the Jordan blocks of size $k$. There are $s_k = t_k - t_{k-1}$ of them.

Step 2: If $k = 1$, stop. Else, apply $T - a$ to the cyclic vectors from the previous step, obtaining $(T-a)v_k^i$ in $\operatorname{null}(T-a)^{k-1}$. Then the key point is that

$$((T-a)v_k^1, \ldots, (T-a)v_k^{s_k}) \text{ is linearly independent mod null}(T-a)^{k-2}.$$

Extend to a basis of $\operatorname{null}(T-a)^{k-1}$ mod $\operatorname{null}(T-a)^{k-2}$, by choosing $(v_{k-1}^1, \ldots, v_{k-1}^{s_{k-1}})$, which are then cyclic vectors for the Jordan blocks of size $k-1$. Since we have extended a list of size $s_k$ to reach a length $t_{k-1} - t_{k-2}$, there are $s_{k-1} = (t_k - t_{k-1}) - (t_{k-1} - t_{k-2})$ of these new cyclic vectors.

Step 3: Repeat Step 2, with $k$ replaced by $k - 1$.

Once the algorithm terminates, we may arrange the chosen basis as follows:

$$\beta = (v_1^1, \ldots, v_1^{s_1}, (T-a)v_2^1, v_2^1, \ldots, (T-a)v_2^{s_2}, v_2^{s_2}, \ldots, (T-a)^{i-1}v_i^1, \ldots v_i^1, \ldots, (T-a)^{i-1}v_i^{s_i}, \ldots, v_i^{s_i}, \ldots)$$

this would put $[T]_\beta^\beta$ into Jordan form with blocks of increasing size down the diagonal.

## An example

Find a Jordan basis for the following operator:

$$T = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Clearly the only eigenvalue is 1. Let $\epsilon = (e_1, \ldots, e_5)$ be the standard basis. First we calculate $\mathrm{null}(T-1)^i$:

$$\begin{aligned} \mathrm{null}(T-1) &= \mathrm{span}(e_1, e_2) \\ \mathrm{null}(T-1)^2 &= \mathrm{span}(e_1, e_2, e_3, e_4) \\ \mathrm{null}(T-1)^3 &= \mathrm{span}(e_1, e_2, e_3, e_4, e_5). \end{aligned}$$

Since $\mathbf{t} = (0, 2, 4, 5, 5, \ldots)$, we have $\mathbf{s} = (0, 0, 1, 1, 0, 0, \ldots)$, so we will have a Jordan form with a block of size 2 and one of size 3. The algorithm proceeds as follows:

Step 1: Pick $v_3^1 = e_5$, it will be a cyclic vector for a Jordan block of size 3.

Step 2: We must extend $(T-1)e_5 = e_1 + e_2 + e_4$ to a basis of $\mathrm{null}(T-1)^2$ mod $\mathrm{null}(T-1)$. We can do this by adding $v_2^1 = e_3$. This will be a cyclic vector for a Jordan block of size 2, and we are done.

The Jordan basis is

$$\beta = ((T-1)v_2^1, v_2^1, (T-1)^2 v_3^1, (T-1)v_3^1, v_3^1) = (e_1, e_3, e_2, e_1 + e_2 + e_4, e_5).$$

Therefore, we have that $T = PJP^{-1}$, where $P = [I]_\beta^\epsilon$ and $J = [T]_\beta^\beta$ is the Jordan form. Explicitly writing $P$ (its columns are the Jordan basis in terms of the standard basis), we have

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$