1. The tangent space T_x to a point $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ has basis $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$, and the cotangent space T_x^* has dual basis $(dx^1, \dots, dx^n)$.

This means that an arbitrary covector is specified by

$$(x^1, \dots, x^n, p_1 dx^1 + \dots + p_n dx^n)$$

The space of all covectors at all points $x \in Q$ $(= \mathbb{R}^n)$ is the <u>cotangent bundle</u> T^*Q, and has coordinates

$$(x^1, \dots, x^n, p_1, \dots, p_n).$$

Notice that the $(p_1, \dots, p_n)$ are determined by the choice of coords $(x^1, \dots, x^n)$. They are "canonically conjugate" to the $(x^1, \dots, x^n)$.

a) let $(\tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n))_{i=1}^n$ be a new coordinate system on M. Determine the conjugate momenta $(\tilde{p}_i)_{i=1}^n$ as a f^n of x^i, p_j.

b) We saw in class that $M = T^*Q$ is equipped with a Poisson bracket $\{,\}: C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ given by

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} \right)$$

prove that $\{f, g\}$ is unaffected if $(\tilde{x}, \tilde{p})$ coordinates are used. This establishes the existence of a canonical Poisson bracket on T^*Q for any manifold Q.

c) Consider $S^2 = \{ x = (x^1, x^2, x^3) \in \mathbb{R}^3 : |x|^2 = 1 \}$

Then its tangent bundle may be written as follows:

$$TS^2 = \{ (x, y) \in \mathbb{R}^3 : |x|^2 = 1 \text{ and } x \cdot y = 0 \}$$

We can identify TS^2 with T^*S^2 by using the inner product: (x,y) defines a linear form on T_xS^2 via $(x,y)((x, y')) = y \cdot y'$.

Now use $(q^1, q^2) = (x^1, x^2)$ as coordinates on the sphere (away from $x^3 = 0$).

Determine the canonically conjugate momenta

(p_1, p_2) as a function of $(x, y) \in TS^2 = T^*S^2$.

d) Write the Spherical pendulum Hamiltonian

$$H = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) + x_3$$

in terms of the coords (q^1, q^2, p_1, p_2)

e) also write the Spherical pendulum angular momentum

$$L = x^2 y_1 - x^1 y_2$$

in coords (q^1, q^2, p_1, p_2), and check $\{H, L\} = 0$ explicitly in these coords.

2. Consider the harmonic oscillator on $Q = \mathbb{R}$ but add a damping term:

$$m\ddot{x} = -kx - \gamma\dot{x}$$

damping force $\gamma \in \mathbb{R}_{>0}$

Describe the trajectories in Phase space.

Is this a Hamiltonian system?

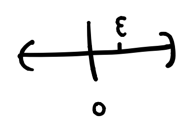
How long does it take for energy to decay from H_1 to $H_0 < H_1$? To $H_1 = 0$?

3. Consider the Harmonic oscillator as a Hamiltonian system on phase space $T^*\mathbb{R} = (\mathbb{R}\times\mathbb{R}, (x,p))$ with Hamiltonian (total energy)

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2$$

Now modify the system by adding a perturbation

$$H(\varepsilon) = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \varepsilon x^3$$



for $\varepsilon > 0$ in some interval about $0 \in \mathbb{R}$.

1. For which energies are there still bound states i.e. trajectories which remain in a bounded region in phase space? Draw a phase portrait.
2. Calculate the 1^{st} nonzero order correction to the amplitude of the $H = 1$ trajectory.

4. (Not to be handed in)

a) Read Woit chapter I.

b) Review Tensor products of vector spaces, and the matrix exponential.

c) Let S, T be linear operators on $\mathbb{C}^n$ and $[S,T]=0$

Prove S stabilizes the eigenspaces of T, and that if S,T are diagonalizable, they are simultaneously diagonalizable.

d) Let $\langle , \rangle$ be the Hermitian inner product on $\mathbb{C}^n$:

$$\langle u, v \rangle = \bar{u}^T v = \sum_{i=1}^{n} \bar{u}_i v_i$$

Let T be a linear operator on $V = \mathbb{C}^n$, and let $W \subset V$ be a T-stable subspace, i.e. $TW \subseteq W$.

Show that $T^* W^\perp \subseteq W^\perp$, where

1) $W^\perp = \{ u \in V : \langle u, w \rangle = 0 \ \forall w \in W \}$

2) T^* is defined by

$$\langle T^* u, v \rangle = \langle u, Tv \rangle.$$

e) Recall that with T as above, when T is <u>normal</u>, i.e. $[T, T^*] = 0$, it may be diagonalized with an orthonormal basis.