Exercise 1 (For your own review, not to be handed in)**.**

1. Let S, T be two linear operators $V \to V$ which *commute*, meaning $ST = TS$. Prove that S stabilizes every eigenspace of T, and prove that if both S and T are diagonalizable, then they are *simultaneously diagonalizable*, i.e., they are both diagonal with respect to a single choice of basis.

2. Let T be a linear operator as above and let $W \subset V$ be T-stable linear subspace. Show that the orthogonal complement $W^\perp$ is T^*-stable. Recall that T^* is the operator defined by the condition
$$\langle T^* v, w\rangle = \langle v, Tw\rangle.$$

3. An operator T is called *normal* when T commutes with T^*. Prove that a normal operator may be diagonalized by an appropriate choice of orthonormal basis.

4. Show that any Hermitian matrix can be diagonalized by a unitary transformation and that the resulting diagonal entries are real.

Exercise 2. Recall that $U(n)$ is the set of $n \times n$ matrices A satisfying

$$\langle Ax, Ay\rangle = \langle x, y\rangle \text{ for all } x, y, \text{ or equivalently } A^*A = \mathbf{1}.$$

1. Show that any unitary matrix can be diagonalized using a unitary change of basis. What are the possible diagonal entries in the result?

2. Show that any unitary matrix may be written as e^X for X a skew-adjoint matrix. Recall that the matrix exponential is given by the convergent power series
$$e^X = \sum_0^\infty \frac{1}{k!} X^k.$$

Exercise 3. Consider the Pauli matrices, a basis for the real vector space of self-adjoint operators on $\mathbb{C}^2$.

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

1. Compute the matrix exponential $e^{-itX}, t \in \mathbb{R}$, for X being each of the Pauli matrices.

2. Define the Hamiltonian operator $H = -B\sigma_1$ for a fixed $B \in \mathbb{R}$ and use it to evolve the state
$$\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
via Schrödinger unitary evolution. Describe the resulting path $\psi(t)$ in state space.

3. Compute the vector $J(t) = (\langle\sigma_1\rangle, \langle\sigma_2\rangle, \langle\sigma_3\rangle)$ of expectation values for the family of states $\psi(t)$. Compare and contrast the evolution of the real 3-vector $J(t)$ with the evolution of the complex 2-vector $\psi(t)$.

Exercise 4. Let $e \in V$ be a fixed unit vector. Does there exist a unitary operator on $V \otimes V$ which takes $v \otimes e$ to $v \otimes v$ for all unit vectors v? Give an example/proof.

Exercise 5. Let $\mathbb{F}_2 = \{0, 1\}$ and let $V = \mathbb{C}^{\mathbb{F}_2} \cong \mathbb{C}^2$ be the Hilbert space of the quantum bit or *qubit*. To any "boolean function" $f : \mathbb{F}_2^n \to \mathbb{F}_2^m$ we may associate a unitary operator U_f on $V^{\otimes n} \otimes V^{\otimes m}$ defined by its value on the basis of decomposable states, which is

$$U_f : |x, y\rangle \mapsto |x, y + f(x)\rangle \,,$$

where the addition is vector addition in $\mathbb{F}_2^m$ and we define, for $x = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$ and $y = (y_1, \ldots, y_m) \in \mathbb{F}_2^m$,

$$|x, y\rangle = |x_1\rangle \otimes \cdots \otimes |x_n\rangle \otimes |y_1\rangle \otimes \cdots \otimes |y_m\rangle \,.$$

1. Show that $U_{f+g} = U_f U_g$.

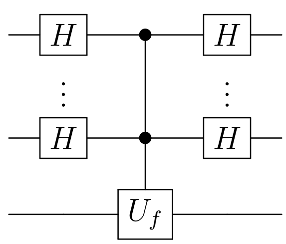
2. Determine the inverse of the composition $U_{f_1} \cdots U_{f_k}$ of unitary operators.

3. Let H be the Hadamard gate; compute the value of $H^{\otimes n}$ on the state $|0\rangle^{\otimes n} \in V^{\otimes n}$.

4. Compute the value of $U_f \cdot (H^{\otimes n} \otimes \mathbf{1})$ on the state $|0\rangle^{\otimes n} \otimes |0\rangle^{\otimes m}$.

5. Suppose we know that $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is either constant or balanced (balanced means that f has value 0 on exactly 2^{n-1} input values). To determine which alternative holds, one would normally have to evaluate f on as many as $2^{n-1} + 1$ values.

   Instead, consider the following circuit defining a unitary operator on $V^{\otimes n} \otimes V$:

   Suppose we feed the circuit with the state $|0 \cdots, 0\rangle \otimes (|0\rangle - |1\rangle)$. Show that by measuring the $n + 1$ output qubits it is possible to determine whether the given function is constant or balanced.

Exercise 6. Particles have an observable called "spin", which may be measured along any direction in 3-dimensional space, by, for example, the Stern-Gerlach experiment. The observed value of the spin along an axis may vary in a certain range which depends on the type of particle. In this exercise we focus on the simplest case, that of a particle where the measured value of the spin along an axis is one of two possible values, usually called "spin up" and "spin down" (relative to the chosen axis direction).

Let $V \cong \mathbb{C}^2$ and consider the Pauli spin observables

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which may be interpreted as the spin of a "spin-$\frac{1}{2}$" particle along the x, y, z axes respectively, in units of $\frac{1}{2}\hbar$.

1. Show that any point $(v_1, v_2, v_3) \in \mathbb{R}^3$ on the unit sphere determines an observable
$$v \cdot \sigma = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$$
which also has eigenvalues ± 1. This is the observable corresponding to the spin of the particle along the axis determined by v.

2. The Pauli observables have been written in a basis $(|0\rangle, |1\rangle)$ of eigenstates for σ_3: that is, $\sigma_3 |0\rangle = |0\rangle$ and $\sigma_3 |1\rangle = -|1\rangle$. Consider the state $|0\rangle + |1\rangle$: what is the probability that it is measured spin up relative to the x, y, z axes? Along which axis does it have a definite measured value for the spin, and why?

3. Suppose we have a system of two spin-$\frac{1}{2}$ particles which are in the entangled state
$$\psi = |00\rangle + |11\rangle \,.$$
Now suppose we only perform measurements on the first particle: what is the probability of obtaining a spin up result, measuring along the x, y, z axes? What about along an arbitrary axis? Prove your result.

4. Now go back to the 1-particle system. Suppose we only have a probabilistic knowledge of the state of the system: we know that the particle has equal odds of being either in state $|0\rangle$ or in state $|1\rangle$. What then would be the probability of measuring spin up in the direction v?

5. Compare these last three results: how is a system which is *entangled* with another system similar to a system where the quantum state itself is described probabilistically?

Exercise 7. Recall that the expectation value of the observable A on the state ψ is defined by
$$E_\psi(A) = \frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle},$$
and represents the probabilistic average result of measuring A repeatedly, given that the system is prepared in state ψ each time before measurement.

The *standard deviation* $\sigma_\psi(A)$, a measure of the deviation of measurements from the expectation value, is then defined as follows:
$$\sigma_\psi(A) = \sqrt{E_\psi\left((A - E_\psi(A))^2\right)}.$$

Two observables A, B define Lie algebra elements iA, iB, with Lie bracket $[iA, iB] = -[A, B]$, defining the observable $i[A, B]$. Prove that
$$\tfrac{1}{2}|E_\psi(i[A, B])| \leq \sigma_\psi(A)\sigma_\psi(B),$$
and provide a heuristic interpretation of the above inequality; how are the predictions of the theory for observables A, B affected by the commutator $i[A, B]$? Note: you may work in a finite-dimensional Hilbert space, though the above result is valid for bounded observables on any Hilbert space.