Reading: Woit Chapter 8, 9, and a brief introduction to Hilbert spaces, I recommend:
`https://www.math.ucdavis.edu/~hunter/book/ch6.pdf`

Exercise 1. Let $V \cong \mathbb{C}^2$ be the standard spin-$\frac{1}{2}$ representation of $SU(2)$, with standard basis vectors (e_0, e_1). The representation $S^2V \otimes S^2V$ decomposes into three irreducible representations; determine which ones. Give an argument using highest weight vectors. Then give explicit bases spanning the three summands, with each basis element expressed in terms of the elements

$$\ell_1 = \tfrac{i}{\sqrt{2}}(e_0 \otimes e_0 - e_1 \otimes e_1), \quad \ell_2 = \tfrac{1}{\sqrt{2}}(e_0 \otimes e_0 + e_1 \otimes e_1), \quad \ell_3 = \tfrac{-i}{\sqrt{2}}(e_0 \otimes e_1 + e_1 \otimes e_0).$$

Recall that this basis for S^2V may be identified with the basis $(X_k = -\frac{i}{2}\sigma_k)_{k=1,2,3}$ for $\mathfrak{su}(2)$, defining an isomorphism between S^2V and the (complexified) adjoint representation.

Exercise 2. The Casimir operator in a representation π of $\mathfrak{su}(2)$ on V is given by the operator (the superscript on L does not mean that it is a square).

$$L^2_\pi = (\pi(X_1))^2 + (\pi(X_2))^2 + (\pi(X_3))^2,$$

where $X_k = -\frac{i}{2}\sigma_k$ as before.

1. Prove that if we change the basis (X_1, X_2, X_3) for $\mathfrak{su}(2)$ using conjugation by a matrix in $SU(2)$, then the operator L^2_π remains unchanged. Hint: Show that L^2_π remains constant if we apply 1-parameter subgroups e^{tX_k} to the basis.

2. Compute the Casimir operator for the action of $SU(2)$ on polynomial functions of $(z_1, z_2) \in \mathbb{C}^2$; verify that the homogeneous polynomials are eigenvectors and determine their eigenvalues.

3. Let L^2_π and L^2_ρ be the Casimir operators for two $SU(2)$ representations π, ρ on vector spaces U, V, and let $L^2_{\pi\otimes\rho}$ be the Casimir for the tensor product representation on $U \otimes V$. Describe explicitly the operator $\frac{1}{2}(L^2_{\pi\otimes\rho} - L^2_\pi - L^2_\rho)$ using the basis (X_1, X_2, X_3). This is the famous Spin-Orbit coupling operator.

Exercise 3. (Vector subspaces of a Hilbert space $\mathcal{H}$) Let ℓ^2 be the Hilbert space of square-summable sequences of complex numbers.

1. Give examples of subspaces of ℓ^2 which a) have infinite dimension and codimension, b) which have finite dimension, and c) which have finite codimension.

2. Give an example of a proper subspace of ℓ^2 which is closed.

3. Give an example of a proper subspace of ℓ^2 which is dense.

4. if $W \subset \mathcal{H}$ is a subspace, show the closure $\overline{W}$ is a subspace, show

$$W^\perp = (\overline{W})^\perp,$$

and show that $W \cap W^\perp = \{0\}$.

5. Show that if $W \subset \mathcal{H}$ is a closed subspace, then W and $\mathcal{H}/W$ naturally inherit a Hilbert space structure.

6. Is it possible that $\overline{W}/W$ be nonzero but finite-dimensional?

Exercise 4. (**The unit sphere in Hilbert space**). Let $S(\mathcal{H}) \subset \mathcal{H}$ be the unit sphere in $\mathcal{H}$.

1. Show that $S(\mathcal{H})$ is closed.

2. Show that a linear map of Hilbert spaces $F : \mathcal{H}_1 \to \mathcal{H}_2$ is continuous if and only if $F(S(\mathcal{H}_1))$ is bounded. (This is why such maps are sometimes called "bounded operators") Show this is equivalent to the inequality
$$||Fv||_{\mathcal{H}_2} \leq C||v||_{\mathcal{H}_1} \quad \forall v \in \mathcal{H}_1, \tag{1}$$
for some constant C independent of v.

3. Suppose that $D \subset \mathcal{H}_1$ is a dense linear subspace and $F : D \to \mathcal{H}_2$ is a continuous linear map. Show that F has a unique extension to a continuous linear map $F : \mathcal{H}_1 \to \mathcal{H}_2$.

4. The operator $\frac{d}{dx}$ is defined on a dense subspace $D \subset L^2(\mathbb{R})$ containing the smooth functions with compact support (meaning that the function vanishes outside some finite interval in $\mathbb{R}$). Show that $\frac{d}{dx}$ is not bounded, that is, that $\frac{d}{dx} : D \to L^2(\mathbb{R})$ is not continuous.

   Despite the fact that $\frac{d}{dx}$ is not defined on all of $L^2(\mathbb{R})$, we refer to it as an operator on $L^2(\mathbb{R})$, keeping in mind that it is defined only on a dense subspace called its *domain*.

Exercise 5. (**The continuous dual**) The operator norm of a continuous linear map $F : \mathcal{H}_1 \to \mathcal{H}_2$ is defined as
$$||F|| := \sup_{v \in S(\mathcal{H}_1)} ||Fv||_{\mathcal{H}_2}.$$
Show that the composition of continuous linear operators is a continuous operation in the operator norm, i.e. for A, B continous linear operators, show
$$||A \circ B|| \leq ||A|| ||B||.$$

Let $\mathcal{H}'$ denote the continuous dual of $\mathcal{H}$, i.e. the space of continuous linear maps $L : \mathcal{H} \to \mathbb{R}$, equipped with operator norm, viewing $\mathbb{R}$ as a Hilbert space.

1. Show that the "dualization map" $v \mapsto v^* = \langle v, \cdot \rangle$ is an injective, norm-preserving continuous linear map $\mathcal{H} \to \mathcal{H}'$.

   The Riesz representation theorem states that the dualization map is an isomorphism of Hilbert spaces.

2. Show that if $F : \mathcal{H}_1 \to \mathcal{H}_2$ is a continous linear operator, then $F^* : \mathcal{H}_2' \to \mathcal{H}_1'$ defined by
$$F^* \mu = \mu \circ F$$
is a continuous linear map. If F is injective, under what conditions is F^* surjective? Show that if F is injective and $\mathrm{Im}(F)$ is dense, then F^* is injective.