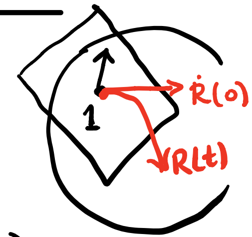
Representations of $SU(2)$ and $SO(3)$



$$SO(3) = \{ R \in \mathbb{R}^{3\times 3} \; : \; RR^T = 1 \}$$

$$\mathfrak{so}(3) = \text{Lie algebra of the Lie group } SO(3)$$
$$= \text{tangent space to } SO(3) \text{ at } 1 \in SO(3)$$

$R(t) \qquad R(0) = 1$

$R(t)\,R(t)^T = 1 \quad \forall t$

$\dot{R}(t)\,R(t)^T + R(t)\,\dot{R}(t)^T = 0$

set $t=0 \Rightarrow \dot{R}(0) + \dot{R}(0)^T = 0$

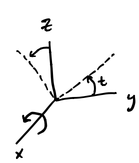
$$\begin{pmatrix} 0 & a & c \\ -a & 0 & b \\ -c & -b & 0 \end{pmatrix}$$

$$= \{ X \in \mathbb{R}^{3\times 3} \; : \; X + X^T = 0 \} = 3\times 3 \text{ skew matrices.}$$

$$= \operatorname{Span}\left(\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{= \Sigma_1}, \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{= \Sigma_2}, \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{= \Sigma_3} \right)$$

Σ_1 = inf. rotation about x-axis

$$e^{t\Sigma_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$



<u>note</u>:

$$\left\{ \begin{aligned} [\Sigma_1, \Sigma_2] &= \Sigma_1\Sigma_2 - \Sigma_2\Sigma_1 = \Sigma_3 \\ [\Sigma_2, \Sigma_3] &= \Sigma_1 \\ [\Sigma_3, \Sigma_1] &= \Sigma_2 \end{aligned} \right\}$$

these three commutators define Lie bracket on $\mathfrak{so}(3)$.

$$SU(2) = \{ U \in \mathbb{C}^{2\times 2} : U\bar{U}^T = 1 \text{ and } \det U = 1 \}$$

$$= \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 + |\beta|^2 = 1 \right\}$$

$\alpha = x_0 + i x_1 \quad \beta = x_2 + i x_3 \qquad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$

$(= S^3 \subset \mathbb{R}^4$ 3-d unit sphere$)$

$\mathfrak{su}(2)$ = Lie algebra of $SU(2)$

$U(t) \quad U(0) = 1 \quad U(t)\overline{U(t)}^T = 1 \quad \det U(t) = 1$

diff. and set $t=0$ $\Rightarrow$ $\begin{cases} \dot{U}(0) + \overline{\dot{U}(0)}^T = 0 & ① \\ \boxed{\left.\frac{d}{dt}\right|_{t=0} \det U(t)} = 0 & ② \end{cases}$

$= \operatorname{Tr} \dot{U}(0)$

① $\dot{U}(0)$ skew adjoint (Lie algebra of the unitary group).

② $U(t) = 1 + Xt + o(t^2)$, $\quad X$ sk. adjoint.

$\wr$

$U(t) \sim \begin{pmatrix} 1+\lambda_1 t & 0 \\ 0 & 1+\lambda_2 t \end{pmatrix} + o(t^2) \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\det U(t) = 1 + (\lambda_1 + \lambda_2)t + o(t^2)$

$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} \det(U(t)) = \lambda_1 + \lambda_2 = \operatorname{Tr} X$

3d$_{\mathbb{R}}$.

$$= \{ X \in \mathbb{C}^{2\times 2} : X + \bar{X}^T = 0 \text{ and } \operatorname{Tr} X = 0 \}.$$

$$= \operatorname{span}\Big(\underbrace{-\tfrac{i}{2}\sigma_1}_{X_1}, \underbrace{-\tfrac{i}{2}\sigma_2}_{X_2}, \underbrace{-\tfrac{i}{2}\sigma_3}_{X_3} \Big) \cong \mathbb{R}^3$$

Self-adjoint

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[X_1, X_2] = X_3$ + c.p. $\qquad [X_2, X_3] = X_1$

$[X_3, X_1] = X_2$.

$\cong \mathfrak{so}(3)$ via correspondence

$E_i \longmapsto X_i$

What is the relation between SU(2) and SO(3)?

Adjoint action Any Lie group G has a natural action (represent.) on its Lie algebra $\mathfrak{g}$. (called the adjoint representation).

Ex.: $SU(2) \ni u$ $\quad x \in su(2)$

$$\boxed{Ad_u(x) = u x u^{-1}}$$

$$\overline{(u x u^{-1})}^T = \overline{u}^{-1\,T}\, \overline{x}^T\, \overline{u}^T \underset{u\overline{u}^T = 1}{=} u\, \overline{x}^T u^{-1} = -\, u x u^{-1}$$

$\Rightarrow \quad u x u^{-1} \in su(2)$ as required

$$\begin{aligned} Ad_{u_1 u_2}(x) &= u_1 u_2 x (u_1 u_2)^{-1} \\ &= u_1 (u_2 x u_2^{-1}) u_1^{-1} \\ &= Ad_{u_1}(Ad_{u_2}(x)) \end{aligned}$$

$$\boxed{\begin{aligned} \pi : SU(2) &\longrightarrow L(su(2)) \\ u &\longmapsto Ad_u \\ \pi(u_1 u_2) &= \pi(u_1) \circ \pi(u_2) \end{aligned}}$$

SU(2) has a 3d representation:

the Adjoint rep'n on $\mathfrak{su}(2)$

a general element in $\mathfrak{su}(2)$ would be

$$aX_1 + bX_2 + cX_3 = \frac{-i}{2}\begin{pmatrix} c & a-ib \\ a+ib & -c \end{pmatrix}$$

$$\det(aX_1 + bX_2 + cX_3) = \frac{-1}{4}\left(-c^2 - a^2 - b^2\right)$$

$$= \frac{1}{4}\left(a^2+b^2+c^2\right) \quad \left(\begin{array}{l}\text{Euclidean} \\ \text{norm} \\ \text{on } \mathbb{R}^3!\end{array}\right)$$

[We can think of a vector $(a,b,c) \in \mathbb{R}^3$ as a vector X in $\mathfrak{su}(2)$, and its Euclidean squared norm is just $\det X$.]

But then $\det(uXu^{-1}) = \det X$ implies that $Ad_u : \mathbb{R}^3 \to \mathbb{R}^3$ preserves Euclidean norm, so is an elt of SO(3)!

Homom. of groups

$$\boxed{\begin{array}{ccc} SU(2) & \xrightarrow{\tau} & SO(3) \\ \rotatebox{90}{$\in$}\, u & \longmapsto & Ad_u \end{array}}$$

Properties of τ:

① τ is surjective: any $R \in SO(3)$ is a rot. by t radians about some axis. Choose basis s.t. axis = z-axis.

wlog we may assume

$$R = e^{t\Sigma_3} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\shortparallel$$

want: elt $U \in SU(2)$ s.t. Ad_U

Compute deriv. of τ

$$D\tau : su(2) \longrightarrow so(3).$$

$X_1 \in su(2)$ generates path $e^{tX_1} \in SU(2)$

acts on $so(3)$ via

$$\tau(e^{tX_1})(X) = Ad_{e^{tX_1}}(X) = e^{tX_1} X e^{-tX_1}$$

$$D\tau(X_1) : X \longmapsto \frac{d}{dt}\Big|_{t=0} e^{tX_1} X e^{-tX_1} = X_1 X - X X_1 = [X_1, X]$$

$$D\tau(X_1) \;:\; X \longmapsto [X_1, X]$$

$$X_1 \longmapsto [X_1, X_1] = 0$$

$$X_2 \longmapsto [X_1, X_2] = X_3$$

$$X_3 \longmapsto [X_1, X_3] = -[X_3, X_1] = -X_2$$

$D\tau(X_1)$ is transf. of $\mathbb{R}^3 = su(2)$ which on basis (X_1, X_2, X_3) has matrix

$$\Sigma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$D\tau(X_2)$ acts on basis of $su(2)$ via.

$$X_1 \longmapsto [X_2, X_1] = -[X_1, X_2] = -X_3$$

$$X_2 \longmapsto [X_2, X_2] = 0$$

$$X_3 \longmapsto [X_2, X_3] = X_1$$

so has matrix
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \Sigma_2$$

$$\boxed{D\tau(X_i) = \Sigma_i \qquad i = 1, 2, 3}$$

Summary: $SU(2) \ni U \overset{\tau}{\longmapsto} Ad_U \in SO(3)$

has derivative $D\tau : \mathfrak{su}(2) \to \mathfrak{so}(3)$

which is an <u>isomorphism</u> of Lie algebras

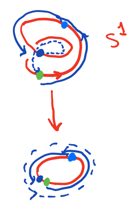
with $D\tau(X_i) = \Sigma_i \qquad i = 1, 2, 3.$

<u>Warning:</u> Just because $D\tau$ isomorphism, doesn't mean τ is!

$$U(1) \xrightarrow{\tau} U(1)$$

$$e^{i\theta} \longmapsto e^{2i\theta}$$

Surjective but not injective: both $e^{i0} = 1$ and $e^{i\pi} = -1$ are sent to 1



① τ is surjective: any $R \in SO(3)$ is a rot. by t radians about some axis. Choose basis s.t. axis = z-axis.

wlog we may assume

$$R = e^{t\Sigma_3} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\shortparallel$$

want: elt $U \in SU(2)$ s.t. Ad_U

to find $U \in SU(2)$ s.t. $\tau(U) = R$,

$$e^{tX_3} = \exp t \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & +\frac{i}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-it/2} & 0 \\ 0 & e^{it/2} \end{pmatrix} = U(t).$$

but notice if $t = 2\pi$ we have

$$e^{2\pi X_3} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$ whereas

$$e^{2\pi \mathcal{E}_3} = \begin{pmatrix} 1 & & \\ & \cos 2\pi & -\sin 2\pi \\ & \sin 2\pi & \cos 2\pi \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$\tau \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

<u>check</u>: $\ker \tau = \pm 1$ in $SU(2)$

$$\boxed{1 \to \{\pm 1\} \longrightarrow SU(2) \longrightarrow SO(3) \to 1}$$

In fact, we saw $SU(2) \cong S^3 \subset \mathbb{R}^4$

and $\tau : SU(2) \to SO(3)$ is quotient

by antipodal map $A : S^3 \to S^3$

$x \longmapsto -x$

i.e. $\boxed{SO(3) = S^3/\mathbb{Z}_2}$

$S^n/\mathbb{Z}_2$ antipodal quotient $= \mathbb{RP}^n$

$\Rightarrow$ $SO(3) = \mathbb{RP}^3$ 3-d real projective space.

Representations of SU(2)

Recall U(1) has irreducible representations labeled by an integer k "weight" "charge"

$$U(1) \xrightarrow{\pi_k} L(\mathbb{C})$$
$$e^{i\theta} \longmapsto [e^{ik\theta}] \quad \text{linear op. on } \mathbb{C}$$

Inside SU(2) we have a (in fact many) copy of U(1).

$H = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ generates a $U(1) \subset SU(2)$

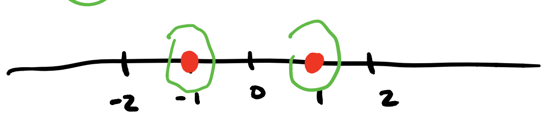
$$e^{i\theta\sigma_3} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \qquad U(1) \hookrightarrow SU(2)$$

$\Rightarrow$ defines a 2d $\mathbb{C}$ rep'n of U(1)

$$\pi(e^{i\theta}) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in L(\mathbb{C}^2)$$

$$\mathbb{C}^2 = \underset{e^{i\theta}}{\underset{\circlearrowleft}{\mathbb{C}}} \oplus \underset{e^{-i\theta}}{\underset{\circlearrowleft}{\mathbb{C}}}$$

decomposes as a sum of irred. rep'ns with weights +1, -1



we have taken the standard represent. $\mathbb{C}^2$ of $SU(2)$ and expressed it in terms of irred rep'ns of a particular $U(1) \hookrightarrow SU(2)$.

Warning: $\mathbb{C}^2$ is decomposable as $U(1)$ rep'n. But irreducible as a rep'n of $SU(2)$.
This means that the other (non-diag) matrices in $SU(2)$ must mix the weight spaces up.

so far we used σ_3 (since it is diagonalized)
look now at the others: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\left\{ \begin{aligned} E &= \frac{\sigma_1 + i\sigma_2}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ F &= \frac{\sigma_1 - i\sigma_2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ H &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \right\} \text{ alternative basis for } su(2) \otimes \mathbb{C}$$

commutation relations :
$$\begin{cases} [H, E] = HE - EH = E - (-E) = 2E \\ [H, F] = -2F \\ [E, F] = H \end{cases}$$

Main calculation (for rep'ns of Lie algebras)

Suppose we have a representation V of $su(2)$

i.e. $\rho \in \pi'$) $\quad \rho : su(2) \to L(V)$

$$\rho([a,b]) = [\rho(a), \rho(b)].$$

(1) Suppose $v \in V$ s.t. $\rho(H)(v) = Hv = kv$

v is a vector with weight k

for the $U(1)$ action $U(\theta) = e^{i\theta H} \circlearrowright V$

Then apply E:

what is weight of Ev ?

$$HEv = (EH + 2E)v$$

$$= E\underbrace{Hv}_{kv} + 2Ev$$

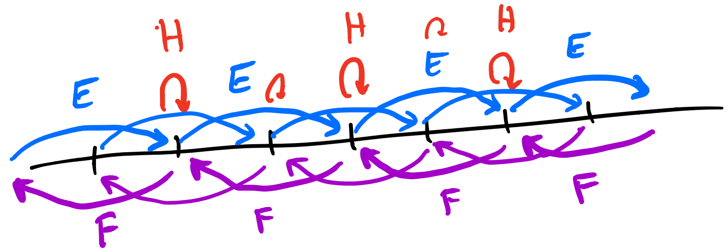
$$= (k+2)Ev$$

If v has wt k, Ev has wt $k+2$

E raises weight by 2.

② apply F: $$HFv = (FH - 2F)v$$
$$= (k-2)Fv$$

F lowers weight by 2

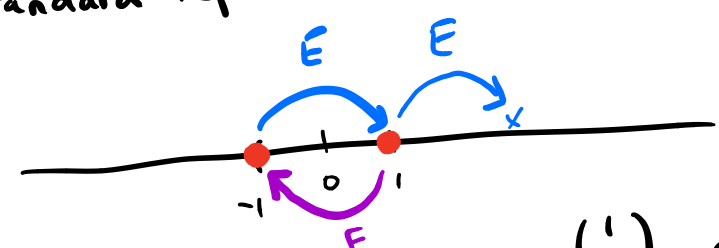


Summarize: analyse any rep'n of SU(2) V

by ① identifying the weights for $U(1) \subset SU(2)$ given by $e^{i\theta H}$

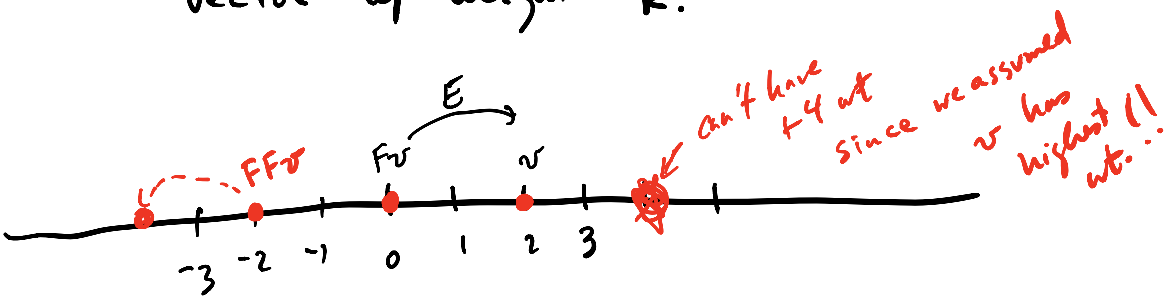
② describe how E, F act on the weight spaces.

Eg: Standard rep'n $SU(2) \curvearrowright \mathbb{C}^2 = \overset{\binom{0}{1}}{\mathbb{C}_{-1}} \oplus \overset{\binom{1}{0}}{\mathbb{C}_1}$



note: If we apply E to a vector $\binom{1}{0}$ get $\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

Def^n a $v \neq 0$ in V_k (weight k subspace) s.t. $Ev = 0$ is called a highest weight vector w/ weight k.



Aside: Build a rep'n of $su(2)$ from data of highest wt vector

Suppose $v \neq 0$ $\quad Hv = 2v \quad$ and $\quad Ev = 0$

Apply F: $\quad Fv \quad$ must have wt $2 - 2 = 0$

$$\begin{cases} Fv \neq 0 \\ HFv = 0 \end{cases}$$

(we can't have $Fv = 0$ because

So far: have v, Fv, what else?

$$\left.\begin{aligned} EFv &= (\cancel{FE} + H)v \\ &= Hv = 2v \end{aligned}\right\} \Rightarrow EFv \in \mathrm{Span}(v, Fv)$$

apply F again: $\quad FFv \neq 0 \quad$ If $FFv = 0$

then $EFFv = 0$

$\quad\;\; \|$

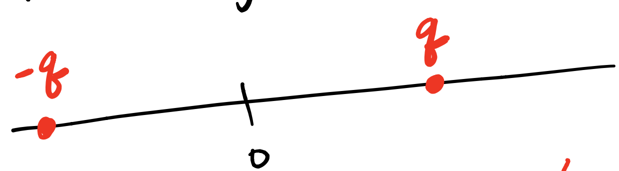
$(FE + H)Fv = 2Fv \neq 0$

So far have $(v, Fv, F^2 v)$ of wts $2, 0, -2$.

Claim: $F(F^2 v) = 0$

$V = \operatorname{Span}(v, Fv, F^2 v)$ is irred. repres.

Why? Weights must be symmetric about zero



If $\exists\, v : Hv = qv$ then $\exists\, v' : Hv' = -qv'$.

$$\pi : SU(2) \longrightarrow L(V) \quad \text{represent.}$$

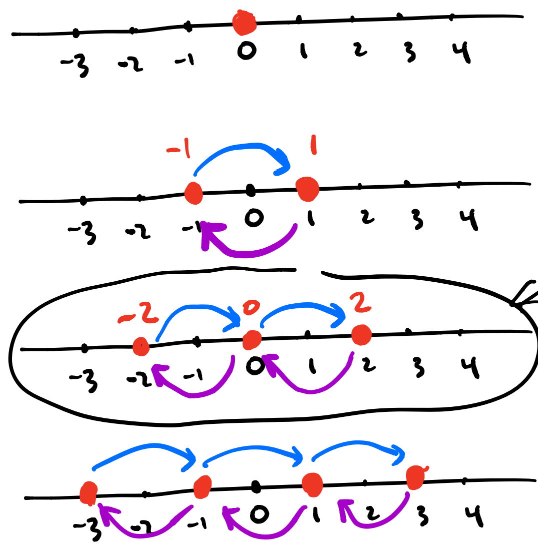
$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \pi(J)$$

$$Ad_J(H) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\pi(J)\, \pi(H)\, \pi(J)^{-1} = -\pi(H)}$$

$\pi(J)$ is isomorphism $V \longrightarrow V$ which exchanges wt $q, -q$.

Thm: every f.dim. irred rep'n of $SU(2)$ is one of the following:



dim		
1	trivial $\mathbb{C}$	$\rho(E)=0$, $\rho(F)=0$, $\rho(H)=0$
2	standard $\mathbb{C}^2$	$\rho(E)=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\rho(F)=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $\rho(H)=\begin{pmatrix}1&\\&-1\end{pmatrix}$
3	usual rot. in 3d.	
4		

there is exactly one irred. rep'n of each dim >0

The 3d rep'n we've already seen:

$\tau: SU(2) \to SO(3) \subset \mathbb{R}^3 \rightsquigarrow \mathbb{C}^3$

$$\tau(e^{i\theta H}) = e^{-2\theta\Sigma_3} = \begin{pmatrix}\cos(2\theta) & \sin 2\theta & 0\\ -\sin 2\theta & \cos 2\theta & 0\\ 0 & 0 & 1\end{pmatrix}.$$

in $\mathbb{C}^3$ this diagonalizes to $\begin{pmatrix}e^{i2\theta} & & \\ & 1 & \\ & & e^{-i2\theta}\end{pmatrix}$ $\begin{matrix}\leftarrow \text{wt } 2\\ \leftarrow \text{wt } 0\\ \leftarrow \text{wt } -2.\end{matrix}$

Concrete description of all these irred. reps of $SU(2)$

Strategy: Build them from standard rep'n $\mathbb{C}^2$ via symmetric tensor products.

V vector space $\quad V \otimes V \supset \mathrm{Sym}^2 V \quad (S^2 V)$

$$\mathrm{Sym}^2 V = \mathrm{Span}(u \otimes v + v \otimes u)$$

$(e_1, \dots, e_n)$ basis for V.

$\mathrm{Sym}^2 V$ has basis $(e_1 \otimes e_1, \dots, e_n \otimes e_n,$
$e_i \otimes e_j + e_j \otimes e_i \quad i < j$).

$$\binom{n}{2} + n = \frac{n(n+1)}{2} = \dim \mathrm{Sym}^2 V.$$

$$V \otimes V \otimes V \supset \mathrm{Sym}^3 V = S_3\text{-invariant subspace}$$

equivalent description of this

$(e_1, \dots, e_n)$ basis of V - view it as variables of a polynomial ring

$$\mathbb{C}[x_1, \dots, x_n] \supset \text{homogeneous polynomials of degree } d = \mathrm{Sym}^d V$$

$$\boxed{\begin{gathered}\mathbb{C}[x_1, \dots, x_n] = \bigoplus_{d=0}^{\infty} \mathrm{Sym}^d V \\ V = \mathrm{span}(x_1, \dots, x_n).\end{gathered}}$$

$V = \mathbb{C}^2$ standard representation of $SU(2)$.

apply natural operations to V $\longrightarrow$ produce new rep'ns.

e.g. ① $V \otimes V$

$$SU(2) \ni U \cdot (v_1 \otimes v_2) = (Uv_1) \otimes (Uv_2)$$

② $\mathrm{Sym}^2 V$

$$U(v_1 \otimes v_2 + v_2 \otimes v_1) = Uv_1 \otimes Uv_2 + Uv_2 \otimes Uv_1$$

③ $\mathrm{Sym}^d V$

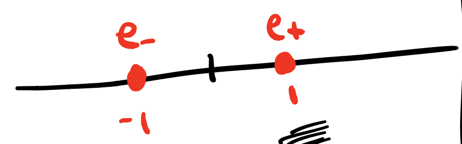
Q.: If $V = \mathbb{C}^2$ and we produce $W = \mathrm{Sym}^d V$

which rep'n in classif. is W?

is it even irreducible?

Ex.: $\mathrm{Sym}^2 V$

$$V = \mathrm{Span}(e_-, e_+)$$



$$\mathrm{Sym}^2 V = \mathrm{Span}\left(e_- \otimes e_-,\ e_- \otimes e_+ + e_+ \otimes e_-,\ e_+ \otimes e_+\right)$$

3d representation.

how to determine if irred? look for highest weight vectors!

$\pi'(H)$ eigenvectors.

$$\begin{cases} \pi(U)(v_1 \otimes v_2) = Uv_1 \otimes Uv_2 & \text{group action} \\ \pi'(H)(v_1 \otimes v_2) = (Hv_1) \otimes v_2 + v_1 \otimes (Hv_2) & \text{Lie algebra action.} \end{cases}$$

$$\pi'(H)(e_+ \otimes e_+) = He_+ \otimes e_+ + e_+ \otimes He_+$$
$$= 2(e_+ \otimes e_+)$$

$$\pi'(E)\,(\underbrace{e_+\otimes e_+}_{\substack{\text{highest}\\ \text{wt v. with weight } 2}}) = \underbrace{\cancel{E e_+}\otimes e_+ + e_+\otimes \cancel{E e_+}}_{=0}$$

$\Rightarrow$ $Sym^2(\mathbb{C}^2)$ is irred rep'n of $SU(2)$ w/ highest wt 2.

Same arg. $\Rightarrow$ $\boxed{Sym^d\,\mathbb{C}^2 \text{ is the } (d+1)\text{-dim}^n \text{ rep'n of } SU(2)}$