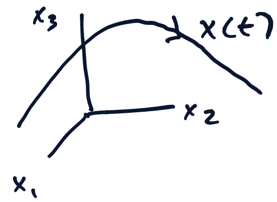
I. Classical mechanics (Hamiltonian formalism).

Motion of a point particle in a force field

position $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

Force field $F(x) = (F_1(x), \dots, F_n(x))$ mass $m \in \mathbb{R}_{>0}$

time $t \in \mathbb{R}$ and trajectory of particle is map



$$x : \mathbb{R} \to \mathbb{R}^n$$

Newton's Law $F(x) = m\ddot{x}$

<u>1^{st} order formalism</u>: define $p = m\dot{x}$ "momentum"

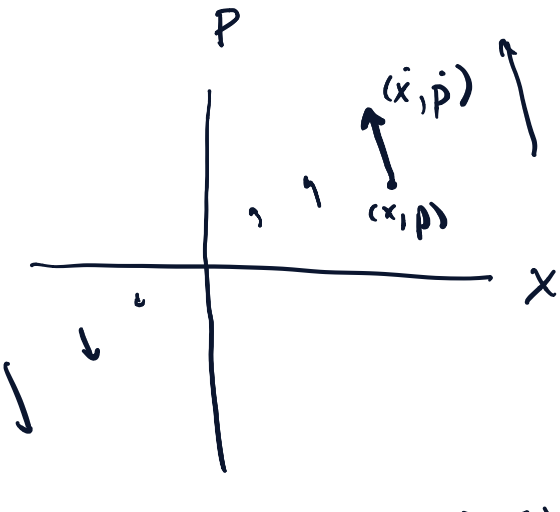
then

$$\underbrace{F(x) = m\ddot{x}}_{\text{2nd order D.Eq.}} \iff \underbrace{\begin{cases} p = m\dot{x} \\ F = \dot{p} \end{cases}}_{\text{system of 1st order D.Eq.}}$$

<u>Main advantage</u>: if we specify $x(t_0)$ and $p(t_0)$ the system tells us $x(t_0 + \Delta t)$ and $p(t_0 + \Delta t)$

it becomes an initial value problem: the <u>state</u> of the system at time t is $(x(t), p(t))$

and the state evolves by solving a 1^{st} order ODE



at any (x, p) the velocity $(\dot{x}, \dot{p})$ is given: $(\dot{x}, \dot{p}) = (\frac{p}{m}, F(x))$

In other words, the 1^{st} order system may be interpreted as a <u>vector field</u> V on the <u>phase space</u> M

vector-valued smooth f^n

space of all positions and momenta

<u>Solving the system</u> $\Longleftrightarrow$ finding an <u>integral</u> <u>curve</u> of V through (x_0, p_0)

(with init. cond (x_0, p_0))

<u>Def^n</u>
An integ. curve of V through (x_0, p_0) is a smooth map

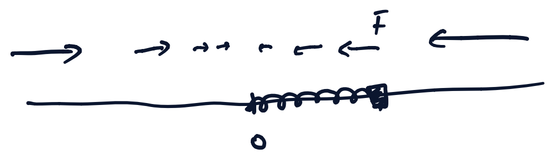
$\gamma : \mathbb{R} \longrightarrow M$ $\quad \gamma(0) = (x_0, p_0)$, $\dot{\gamma}(t) = V(\gamma(t))$.

Ex.: 1 The harmonic oscillator

configuration space $(\mathbb{R}, x)$

mass $m \in \mathbb{R}_{>0}$

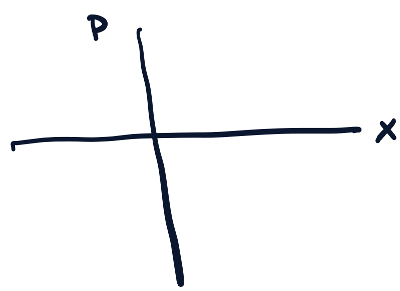
Force $F(x) = -kx$ $k \in \mathbb{R}_{>0}$ "spring constant"



momentum $p = m\dot{x} \in \mathbb{R}$

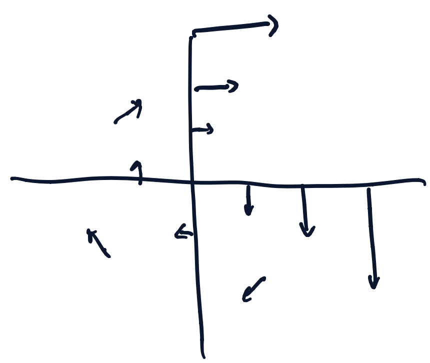
phase space

$\mathbb{R} \times \mathbb{R}$



v. field $V(x,p) = \left(\frac{p}{m}, -kx\right) =$

let $m = k = 1$:



Find integral curves : especially easy since V is a linear v. field i.e. $V(\lambda x, \lambda p) = \lambda V(x,p)$.

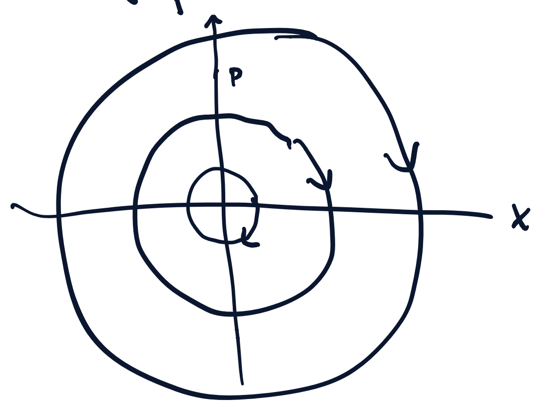
$$\begin{pmatrix} p/m \\ -kx \end{pmatrix} = \begin{bmatrix} 0 & m^{-1} \\ -k & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \underset{\begin{pmatrix} k=1 \\ m=1 \end{pmatrix}}{=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}$$

$$\dot{\gamma} = A\gamma \qquad A^2 = -\left(\frac{k}{m}\right) I = -\omega^2 I$$

$$\boxed{\gamma(t) = e^{tA}\gamma(0)} \qquad \boxed{(\omega^{-1}A)^2 = -I}$$

$$e^{tA} = (\cos t) I + (\sin t) A$$

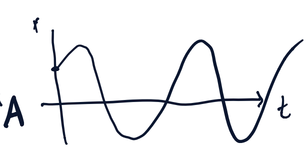
$$\Rightarrow \quad \gamma(t) = \begin{pmatrix} x(0)\cos t + p(0)\sin t \\ p(0)\cos t - x(0)\sin t \end{pmatrix}.$$



$$e^{tA} = e^{\left(\sqrt{\frac{k}{m}}\,t\right)\left(\frac{1}{\sqrt{k/m}}A\right)} = e^{\omega t(\omega^{-1}A)}$$

what does the resulting motion look like?

$$x(t) = x(0)\cos t + \dot{x}(0)\sin t$$



In general $e^{tA} = \cos(\omega t) I + \sin(\omega t)\omega^{-1}A$

$$\Rightarrow \quad x = \cos(\omega t)\, x(0) + \omega^{-1}\sin(\omega t)\,\dot{x}(0).$$

the period is $\boxed{\frac{2\pi}{\omega} = T}$ $\qquad \omega^2 = \frac{k}{m}.$

Observe that we have a __conserved quantity__: a function on phase space which is inv't.

$$H = \tfrac{1}{2} k x^2 + \tfrac{1}{2} m^{-1} p^2 \overset{k=m=1}{=} \tfrac{1}{2}(x^2+p^2)$$

$$A^T \Lambda + \Lambda A = \begin{pmatrix} & m^{-1} \\ -k & \end{pmatrix}^T \begin{pmatrix} k & \\ & m^{-1} \end{pmatrix} + \begin{pmatrix} k & \\ & m^{-1} \end{pmatrix} \begin{pmatrix} & m^{-1} \\ -k & \end{pmatrix} = 0$$

This conserved quantity is called the __total energy__ or Hamiltonian.

$$H = V + T \qquad \text{potential + kinetic}$$

$$V = \tfrac{1}{2} k x^2 \qquad \text{s.t. } -\frac{\partial}{\partial x} V = F$$

$$T = \tfrac{1}{2} m^{-1} p^2$$

Geometric interpret:

$Q = \mathbb{R}^n$ configuration space.

$V : Q \longrightarrow \mathbb{R}$ Potential.

$-dV = F$ force 1-form

$g(\cdot,\cdot)$ Metric i.e. pos-def symm. bilinear form on TQ

$g(\dot{x},\dot{x}) = \quad m(\dot{x})^2$

$g : TQ \longrightarrow T^*Q$

$\frac{\partial}{\partial x^i} \longmapsto g_{ij}\, dx^j$

in this case $\frac{\partial}{\partial x} \longmapsto m\, dx$ $\qquad [g^{ij}] = [g_{ij}]^{-1}$

$g^{-1} : T^*Q \longrightarrow TQ$

$dx^i \longmapsto g^{ij} \frac{\partial}{\partial x^j}$

$dx \longmapsto m^{-1} \frac{\partial}{\partial x}$ in this case.

$g^{-1} : T^*Q \times T^*Q \longrightarrow \mathbb{R}$

$(p\,dx, p\,dx) \longmapsto m^{-1} p^2$

So Newton's Law: $\quad \gamma : \mathbb{R} \to Q$

$F = g\, \ddot{\gamma}$

Let V be a vector field on M

$$V = \sum V^i \frac{\partial}{\partial x^i} \qquad V^i \in C^\infty(M).$$

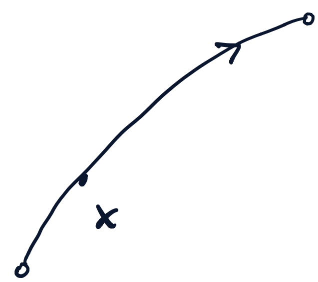
V gives rise to a flow:

through each point $x \in M$ we have!

maximal integral curve $\gamma : (-\varepsilon_-, \varepsilon_+) \longrightarrow M$

$\varepsilon_\pm > 0$

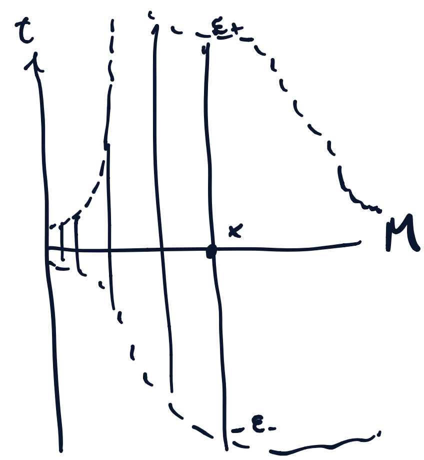
s.t. $\begin{cases} \dot\gamma(t) = V(\gamma(t)) \\ \gamma(0) = x \end{cases}$



If we put all these together we get a flow

$$\varphi_V : U \longrightarrow M$$

$$M \times \{0\} \subset U \underset{\text{open}}{\subset} M \times \mathbb{R}$$



If φ_V is defined on $M \times \{t\}$

then $\varphi_V^t = \varphi_V \circ i_t : M \longrightarrow M$

is a diffeo. onto its image.

It is sometimes useful to write this as

$$\varphi_V^t = e^{tV}$$ due to fact

$$\text{"}\frac{d}{dt}\left(e^{tV}(x)\right) = V\left(e^{tV}(x)\right)\text{"}$$

better is to determine how it acts on f^{ns}

$$\frac{d}{dt}\left((\varphi_V^t)^* f\right) = V\left((\varphi_V^t)^* f\right).$$

so $$f \circ e^{tV} = e^{tL_V} f$$
$$= f + tL_V f + \frac{t^2 L_V^2 f}{2!} + \cdots$$

Taylor series

$$\boxed{(\varphi_V^t)^* f = f + tL_V f + \frac{t^2}{2!} L_V^2 f + \frac{t^k}{k!} L_V^k f + R}$$

R C^∞ defined on nbhd of $M \times \{0\}$

whose order $\leq k$ terms vanish.

<u>Coordinates:</u> $(x^1, \dots, x^n)$ on Q

$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ basis for TQ

$dx^1, \dots, dx^n$ dual basis for T^*Q

$$F = F_1 dx^1 + \dots + F_n dx^n \in \Omega^1$$

$$g = \sum g_{ij}\, dx^i \otimes dx^j$$ bilinear form on TQ.

g defines a map $TQ \longrightarrow T^*Q$

$$\frac{\partial}{\partial x^i} \longmapsto g_{ij}\, dx^j$$

general point in T^*Q is

$$\left((x^1, \dots, x^n),\ p_1 dx^1 + \dots + p_n dx^n\right)$$

so $(x^1, \dots, x^n, p_1, \dots, p_n)$ are coords on T^*Q

(p_i) are "canonically conjugate" to (x^i).

Important observation:

the 1-form $\sum_{i=1}^{n} p_i \, dx^i = \theta$

and 2-form $d\theta = \sum_{i=1}^{n} dp_i \wedge dx^i$

are indep. of coord transform:

$\tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n)$ change of coords

$$d\tilde{x}^i = \sum_j \frac{\partial \tilde{x}^i}{\partial x^j} dx^j$$

$$\tilde{p}_i \, d\tilde{x}^i = \tilde{p}_i \frac{\partial \tilde{x}^i}{\partial x^j} dx^j$$

$$= p_k \, dx^k$$

$$p_k = \sum_i \tilde{p}_i \frac{\partial \tilde{x}^i}{\partial x^k}$$

$$\tilde{p}_i \, d\tilde{x}^i = p_k \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial \tilde{x}^i}{\partial x^l} dx^l = p_k \, dx^k$$

Hamiltonian formalism

The phase space $M = T^*Q$
has intrinsic str. of symplectic mfld
indep. of the forces acting.

$$\omega = dp \wedge dq \quad \in \Omega^2(M)$$

- skew $\omega(X,Y) = -\omega(Y,X)$
- closed $d\omega = 0$
- nondeg $TM \xrightarrow{\omega} T^*M$, $X \mapsto i_X\omega$ is isom.

This gives rise to the following str:

for any function $f \in C^\infty(M)$, $df \in \Gamma(M, T^*M)$

and $-\omega^{-1}(df) = X_f \in \Gamma(M, TM)$ is a vector field.

$$C^\infty(M) \xrightarrow{\omega^{-1}d} \mathfrak{X}(M)$$

Somewhat similar to the gradient of a f^n (defined using metric)

but s.t.

① $X_f(f) = i_{X_f} df = \omega^{-1}(df, df) = 0$

so X_f is <u>tangent</u> to <u>level sets</u> of f

i.e. f is <u>conserved</u> by <u>flow</u> of X_f.

② $\mathcal{L}_{X_f}\omega = d i_{X_f}\omega = d(\omega(\omega^{-1}df)) = d^2 f = 0$

so X_f preserves the background symp. structure.

Conclusion: (M, ω) gives a mechanism for converting a f^n $f \in C^\infty(M)$ into a v.field $X_f \in \mathfrak{X}(M)$ which preserves f and ω. It produces symmetries with desired conservation law.

Furthermore: we have a Lie bracket on $\mathfrak{X}(M)$ and this actually lifts to $C^\infty(M)$: we have a "Poisson bracket" $\{f, g\}$ on $f, g \in C^\infty(M)$.

such that
$$X_{\{f,g\}} = [X_f, X_g].$$

Def^n:
$$\{f, g\} = -\omega^{-1}(df)(dg) = -\omega^{-1}(df, dg)$$
$$= X_f(g) = -X_g(f).$$

In coords $x^1, \dots, x^n$ $p_1, \dots, p_n$ for $(x, p_i dx^i) \in T^*M$

$\omega = \sum_i dp_i \wedge dx^i$ and $\omega : \begin{cases} \partial_{x^i} \longmapsto -dp_i \\ \partial_{p_i} \longmapsto dx^i \end{cases}$

$$X_f = \sum_i \left(\frac{\partial f}{\partial x^i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial x^i} \right)$$

$$\{f, g\} = X_f(g) = \sum_i \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i} \right)$$

Prop: $\{,\}$ is
- skew
- Leibniz $\{f, gh\} = \{f, g\}h + g\{f, h\}$.
- Jacobi $\{\{f, g\}, h\} = \{f\{g, h\}\} - \{g, \{f, h\}\}$.

this means it is a <u>Lie algebra</u> compatible w/ comm. alg. str $\Rightarrow$ Poisson Algebra

And the map $\left(C^\infty(M), \{,\}\right) \xrightarrow{\omega^{-1}d} \left(\mathfrak{X}(M), [,]\right)$

is a Lie algebra homomorphism.

Review: (I) functions $C^\infty(M)$ are "physical observables"

(II) any function f defines a dynamical system

- v.field $X_f = \omega^{-1} df$
- acts on fns $X_f(g) = \{f, g\}$.

so that flow is

$$\boxed{\dot{g} = \{f, g\}}$$

Key properties: 1) $\dot{f} = \{f, f\} = 0$. conserved.

(anything poisson comm. w/ f is conserved!).

2) flow doesn't affect $\{,\}$ itself (jacobi).

eg. – $f \in C^\infty(Q)$ $\quad \pi^* f \in C^\infty(T^*Q)$

$x^1, \dots, x^n$ $\quad \pi^* x^i \in C^\infty(T^*Q)$ "position"

– conjugate coords $p_1, \dots, p_n$ "momenta" $\quad p_i \in C^\infty(T^*Q)$

these are special since $x^1, \dots, x^n$ defines a

commuting subalgebra: $\{x^i, x^j\} = 0$

for $\quad Q = \mathbb{R} \qquad M = T^*Q = \mathbb{R} \times \mathbb{R} \ni (x,p)$

$$\omega = dp \wedge dx$$

$$H = \pi^* V + E_g \qquad \boxed{\begin{matrix} E_g(x, \xi) \\ \shortparallel \\ \frac{1}{2} g_x^{-1}(\xi, \xi) \end{matrix}}$$

$$= \frac{1}{2}(x^2 + p^2) \qquad = \frac{1}{2}\left(k x^2 + \frac{p^2}{m}\right)$$

$$-\omega^{-1} dH = \omega^{-1}(x\,dx + p\,dp)$$

$$= -x \frac{\partial}{\partial p} + p \frac{\partial}{\partial x}$$

The <u>energy</u> function is precisely the one whose associated <u>symmetry</u> is the time evolution.

how does the state evolve ?

state = $(x, p) \in M$

$$\left.\begin{aligned} \dot{x} &= \{H, x\} = \frac{\partial H}{\partial p} \\ \dot{p} &= \{H, p\} = -\frac{\partial H}{\partial x}. \end{aligned}\right\} \text{Hamilton's equations}$$

in our example $\begin{cases} \dot{x} = m^{-1} p \\ \dot{p} = -k x \end{cases}$

<u>Note:</u> Here we have considered 3 observables

x, p, H

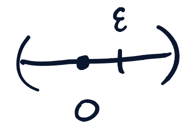
$\{H, x\} = p$

$\{H, p\} = -x$

$\{p, x\} = 1$

Modify H now to

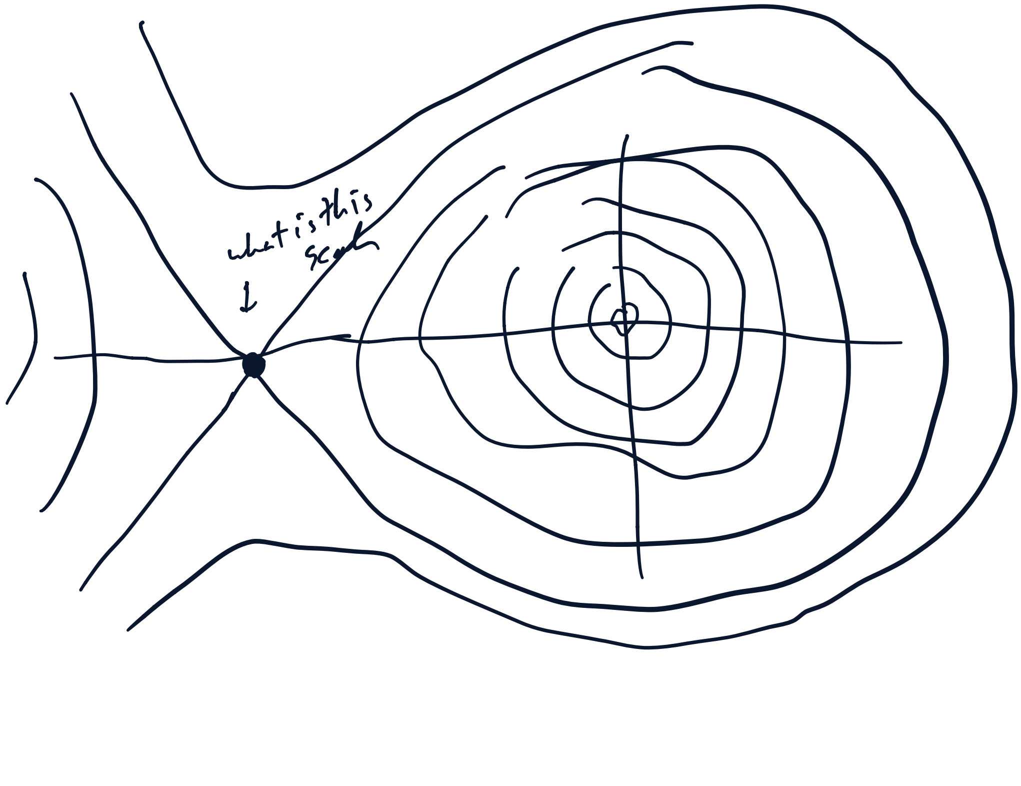
$$H_\varepsilon = \frac{1}{2}\left(x^2 + p^2 + \varepsilon x^3\right)$$



we may ask what happens to, e.g., the oscillation of energy 1

what happens to level sets

$\varepsilon = 0$



$\varepsilon > 0.$

Spherical Pendulum

$$S^2 = \{ (x^1, x^2, x^3) \in \mathbb{R}^3 : \sum x_i^2 = 1 \}.$$

$$T^*S^2 = \{ (x, y) \in T^*\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3 : \|x\|^2 = 1 \text{ and } x \cdot y = 0 \}$$

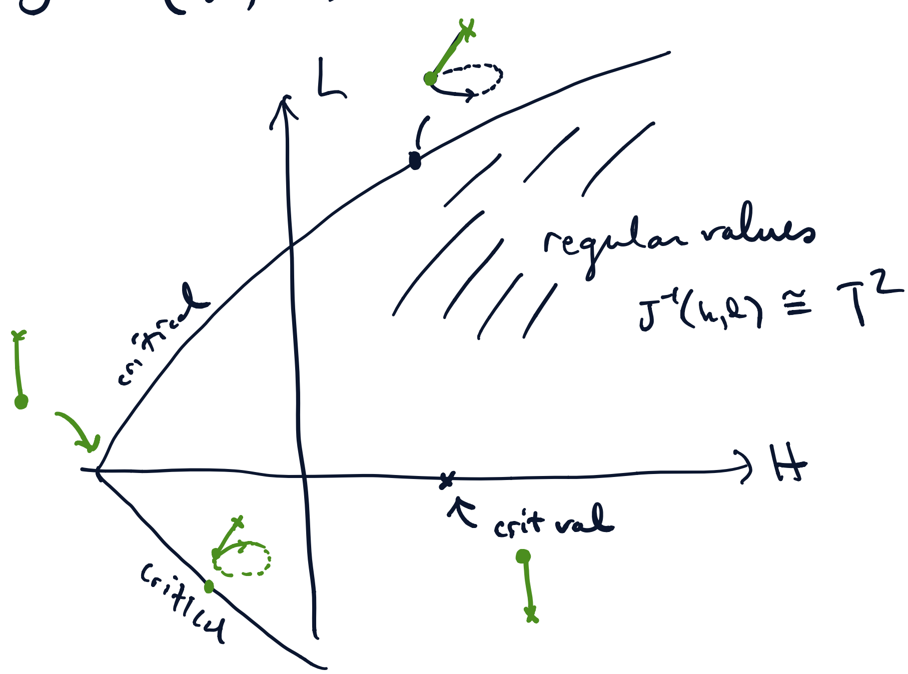
$\boxed{H = \frac{1}{2} y \cdot y + x^3}$ restricted to T^*S^2

$\boxed{L = x^2 y_1 - x^1 y_2}$ generates rot. about x^3 axis

$\{H, L\} = 0$

Energy-Momentum mapping (Moment map)

$$J = (H, L) : T^*S^2 \to \mathbb{R}^2$$



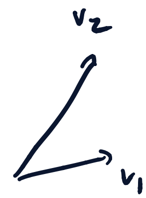
Regular value of J :

$T_{h,\ell} = J^{-1}(h,\ell) =$ • compact smooth 2-mfld.

• X_H and X_J commute,

$X_H \wedge X_J \big|_{T_{(h,\ell)}}$ nonvanishing

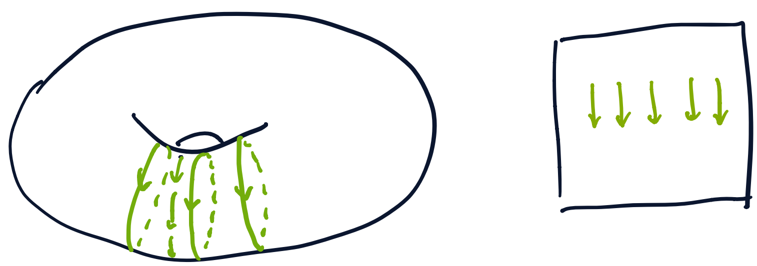
So we get a $\mathbb{R}^2 \times T \longrightarrow T$



$v_1 \wedge v_2 \neq 0.$

$\Rightarrow T$ diffeo. to $\mathbb{R}^2 / \mathbb{Z}v_1 + \mathbb{Z}v_2 \cong T^2.$

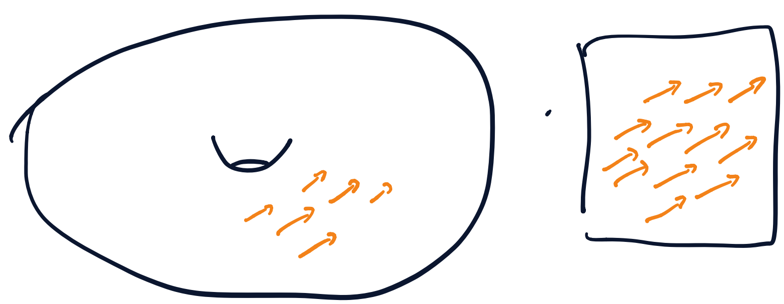
$J^{-1}(h,\ell) \simeq$



X_L

X_L is an S^1 action

X_H is not



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