Recall Laws of QM

1) state is $[v]$ $v \in V$ complex vector space
with Hermitian inner product $\langle , \rangle$

2) Observables are self-adjoint operators $X : V \to V$

$$\langle Xu, v\rangle = \langle u, Xv\rangle$$

and there is a distinguished observable H "the Hamiltonian"

3) State evolves in time via Schrödinger eqn

$$\frac{d}{dt} v = -\frac{i}{\hbar} H v$$

i.e. $v(t) = U(t)\, v(0)$ where $U(t) = e^{-\frac{i}{\hbar} H t}$

is unitary i.e. $\langle U(t)\, a, U(t)\, b\rangle = \langle a, b\rangle$.

4) Measurement (stated for V finite dimensional)

i) The result of measuring observable X must be one of the eigenvalues of X

ii) If P_λ is the orthogonal proj. to λ-eigensp,
then the probability of obtaining λ
is

$$\frac{\| P_\lambda v \|^2}{\| v \|^2} = \frac{\langle v, P_\lambda v\rangle}{\langle v, v\rangle}$$

iii) After measurement with result λ, state is $[P_\lambda v]$.

- A self-adjoint operator has an ON basis of eigenvectors

$$X = \sum x_n P_n \qquad P_n = \text{orthog. proj. to } x_n \text{ eigensp.}$$

$$P_n^2 = P_n \qquad P_n^* = P_n \qquad P_n P_m = 0 \ n \neq m.$$

- orthog. proj. to 1-d space spanned by v is

$$\left(w \longmapsto \frac{\langle v, w \rangle}{\langle v, v \rangle} v \right) = \frac{|v\rangle \langle v|}{\langle v, v \rangle}$$

$\langle v|$ means $l_v = \langle v, - \rangle$ so this is $\dfrac{v \otimes l_v}{\|v\|^2}$

<u>Prop:</u> The expectation value of the observable X on the state $[v]$ is

$$\langle X \rangle = \sum_{\substack{\text{possible} \\ \text{values } \lambda}} \lambda \cdot (\text{Probability of obt. } \lambda)$$

$$= \sum x_n \frac{\langle v, P_n v \rangle}{\langle v, v \rangle} = \frac{\langle v, Xv \rangle}{\langle v, v \rangle}$$

$$\boxed{\langle X \rangle = \frac{\langle v | X | v \rangle}{\langle v, v \rangle}}$$

Combining systems

1. **Classical:** Hamiltonian systems

$(M_1, \{,\}_1, H_1)$ and $(M_2, \{,\}_2, H_2)$

can be combined:

$(M_1 \times M_2, \quad \{,\}, \quad H_1 + H_2)$

where in coords (u_i) for M_1, (v_i) for M_2
$$\begin{cases} \{u_i, u_j\} = \{u_i, u_j\}_1 \\ \{v_i, v_j\} = \{v_i, v_j\}_2 \\ \{u_i, v_j\} = 0. \end{cases}$$

2. **Quantum:** Combining Quantum systems

$(V_1, \langle,\rangle_1, H_1)$ and $(V_2, \langle,\rangle_2, H_2)$

gives $\boxed{(V_1 \otimes V_2, \quad \langle,\rangle, \quad H_1 \otimes 1 + 1 \otimes H_2)}$

(✶✶)

① Aside on the tensor product

$V_1 \otimes V_2$ is the vector space spanned by bilinear expressions $u \otimes v$ where $u \in V_1$, $v \in V_2$.

"bilinear expression" means that

$$(\lambda u_1 + u_2) \otimes v = \lambda (u_1 \otimes v) + u_2 \otimes v$$

and similarly for $u \otimes (\lambda v_1 + v_2)$.

If $(e_i)_{i=1}^{n}$, $(f_j)_{j=1}^{m}$ are bases for V, W

then $(e_i \otimes f_j)_{i=1,\dots,n,\, j=1,\dots,m}$ is a basis for $V \otimes W$

$$\Rightarrow \quad \dim (V \otimes W) = (\dim V)(\dim W)$$

<u>example</u>: If $|0\rangle$ and $|1\rangle$ are basis for V

then $V \otimes V \otimes V$ would have basis

$$\left(|0\rangle \otimes |0\rangle \otimes |0\rangle,\ |0\rangle \otimes |0\rangle \otimes |1\rangle,\ \dots,\ |1\rangle \otimes |1\rangle \otimes |1\rangle \right)$$

which we can write as $(|000\rangle, \dots, |111\rangle)$

labeling each basis vector by a binary number

from 0 to $111 = 8$ $\Rightarrow$ $\dim V^{\otimes 3} = 8$

(2) <u>Combining $\langle , \rangle_1$, $\langle , \rangle_2$</u> :

$$\langle u \otimes v , u' \otimes v' \rangle = \langle u, u' \rangle_1 \langle v, v' \rangle_2$$

$$\langle 101 | 111 \rangle = 1 \cdot 0 \cdot 1 = 0$$

(3) There is a natural inclusion

$$U(V) \times U(W) \longrightarrow U(V \otimes W) :$$

$$(u , u') \longmapsto u \otimes u',$$

where

$$(u \otimes u')(a \otimes b) = ua \otimes u'b$$

So we can act only on V via

$$(u \otimes 1)(a \otimes b) = ua \otimes b$$

- This corresponds physically to exerting forces on only <u>one component</u> of a combined system.
- The operation $(u, u') \mapsto (u \otimes u')$ is the "Kronecker product"

Kronecker product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \left(\begin{array}{c|c} a\begin{pmatrix} e & f \\ g & h \end{pmatrix} & b\begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ \hline c\begin{pmatrix} e & f \\ g & h \end{pmatrix} & d\begin{pmatrix} e & f \\ g & h \end{pmatrix} \end{array}\right)$$

Bases: (e_1, e_2) $\quad (f_1, f_2)$ $\quad (e_1 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_1, e_2 \otimes f_2)$

If we differentiate this:

$$\frac{d}{dt}\Big|_{t=0} \left(e^{itX} \otimes e^{itX'} \right) = i\left(X \otimes 1 + 1 \otimes X' \right)$$

So we see that there is a compatible inclusion.

$$\mathrm{Obs}(V_1, \langle,\rangle_1) \times \mathrm{Obs}(V_2, \langle,\rangle_2) \longrightarrow \mathrm{Obs}(V_1 \otimes V_2, \langle,\rangle)$$

$$\boxed{(X, X') \longmapsto X \otimes 1 + 1 \otimes X'}$$

This completes the justification of

$(\star\star)$

Decomposable states

$$\mathbb{P}(V) \times \mathbb{P}(W) \hookrightarrow \mathbb{P}(V \otimes W) \quad \text{Segre embedding}$$

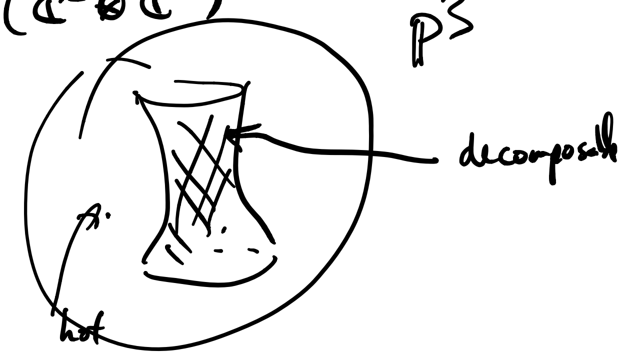
$$([u_0, \dots, u_n], [v_0, v_1, \dots, v_m]) \longmapsto [u_0 v_0, u_0 v_1, \dots, u_i v_j, \dots, u_n v_m]$$

$$\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$$

$$\mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^2) \longrightarrow \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$$

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

$\mathbb{P}^3$



$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \otimes \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$$

$$= \begin{array}{r} x_1 = \\ x_2 = \\ x_3 = \\ x_4 = \end{array} \begin{pmatrix} u_1 u_2 \\ u_1 v_2 \\ v_1 u_2 \\ v_1 v_2 \end{pmatrix} \in \mathbb{C}^4$$

$$x_1 x_4 = x_2 x_3$$

Measurement on Combined System

Consider two copies of $(\mathbb{C}^2, E)$ $\quad E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\left(\mathbb{C}^2 \otimes \mathbb{C}^2, H = \underbrace{E \otimes 1}_{E_1} + \underbrace{1 \otimes E}_{E_2}\right)$$

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & & \\ 0 & 1 & & 0 \\ & & 0 & 0 \\ 0 & & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 2 \end{pmatrix}$$

Eigensp of E_1 are $|0\rangle \otimes \mathbb{C}^2$ and $|1\rangle \otimes \mathbb{C}^2$

If we measure E_1 i.e. "energy of first switch"

on a general state

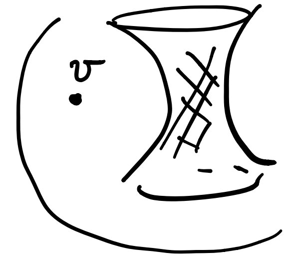
$$v = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

we will get $\begin{cases} 0 & \text{with probability } \dfrac{|a_{00}|^2 + |a_{01}|^2}{|v|^2} \\ 1 & \text{w/ prob. } \dfrac{|a_{10}|^2 + |a_{11}|^2}{|v|^2} \end{cases}$

and if the result is 0 then the resulting state

after measurement is $a_{00}|00\rangle + a_{01}|01\rangle = |0\rangle \otimes (a_{00}|0\rangle + a_{01}|1\rangle)$

Entanglement



Consider the state

$v = |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle = |00\rangle + |11\rangle$

If we measure E_1 we get $\begin{cases} 0 & \text{prob. } 1/2 \\ 1 & \text{prob } 1/2 \end{cases}$

but if we obtain 0 for E_1, the resulting state will be $|00\rangle$, and hence $E_2 |00\rangle = 0$, meaning E_2 is 0 w/ prob. 1.

Similarly if the result of E_1 is 1, E_2 is forced to be 1

So whatever $Meas_{E_1}(v\rangle$ is, it forces E_2 to be the same. $\Rightarrow$ the particles are in an entangled state.