1 Manifolds

A manifold is a space which looks like $\mathbb{R}^n$ at small scales (i.e. “locally”), but which may be very different from this at large scales (i.e. “globally”). In other words, manifolds are made by gluing pieces of $\mathbb{R}^n$ together to make a more complicated whole. We want to make this precise.

1.1 Topological manifolds

Definition 1.1. A real, n-dimensional *topological manifold* is a Hausdorff, second countable topological space which is locally homeomorphic to $\mathbb{R}^n$.

“Locally homeomorphic to $\mathbb{R}^n$” simply means that each point p has an open neighbourhood U for which we can find a homeomorphism $\varphi : U \longrightarrow V$ to an open subset $V \subset \mathbb{R}^n$. Such a homeomorphism φ is called a *coordinate chart* around p. A collection of charts which cover the manifold is called an *atlas*.

We now give examples of topological manifolds. The simplest is, technically, the empty set. Then we have a countable set of points (with the discrete topology), and $\mathbb{R}^n$ itself, but there are more:

Example 1.2 (open subsets)**.** Any open subset $U \subset M$ of a topological manifold is itself a topological manifold, where the charts are simply restrictions $\varphi|_U$ of charts φ for M. For instance, the real $n \times n$ matrices $\mathrm{Mat}(n, \mathbb{R})$ form a vector space isomorphic to $\mathbb{R}^{n^2}$, and contain an open subset

$$GL(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \ : \ \det A \neq 0\}, \tag{1}$$

known as the general linear group, which is a topological manifold.

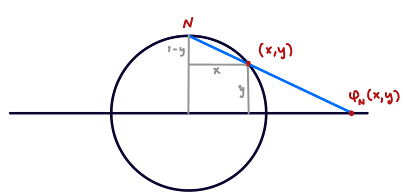
Example 1.3 (Circle)**.** The circle is defined as the subspace of unit vectors in $\mathbb{R}^2$:

$$S^1 = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}.$$

Let $N = (0, 1)$ and $S = (0, -1)$. Then we may write S^1 as the union $S^1 = U_N \cup U_S$, where $U_N = S^1 \backslash \{N\}$ and $U_S = S^1 \backslash \{S\}$ are equipped with coordinate charts φ_N, φ_S into $\mathbb{R}^1$, given by the “stereographic projections” from the points S, N respectively

$$\varphi_N : (x, y) \mapsto (1 - y)^{-1} x, \tag{2}$$

$$\varphi_S : (x, y) \mapsto (1 + y)^{-1} x. \tag{3}$$



By taking products of coordinate charts, we obtain charts for the Cartesian product of manifolds. Hence the Cartesian product of manifolds is a manifold.

Example 1.4 (n-torus)**.** $S^1 \times \cdots \times S^1$ is a topological manifold (of dimension given by the number n of factors), with an atlas consisting of the 2^n charts given by all possible n-fold products of the charts φ_N, φ_S defined above.

The circle is a 1-dimensional sphere; we now describe general spheres.

Example 1.5 (Spheres)**.** The n-sphere is defined as the subspace of unit vectors in $\mathbb{R}^{n+1}$:

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \ : \ \sum x_i^2 = 1\}.$$

Let $N = (1, 0, \ldots, 0)$ be the north pole and let $S = (-1, 0, \ldots, 0)$ be the south pole in S^n. Then we may write S^n as the union $S^n = U_N \cup U_S$, where $U_N = S^n \backslash \{N\}$ and $U_S = S^n \backslash \{S\}$ are equipped with coordinate charts φ_N, φ_S into $\mathbb{R}^n$, given by the "stereographic projections" from the points S, N respectively

$$\varphi_N : (x_0, \vec{x}) \mapsto (1 - x_0)^{-1}\vec{x}, \tag{4}$$

$$\varphi_S : (x_0, \vec{x}) \mapsto (1 + x_0)^{-1}\vec{x}. \tag{5}$$

Remark 1.6. We have endowed the sphere S^n with a certain topology, but is it possible for another topological n-manifold X to be homotopy equivalent to S^n *without* being homeomorphic to it? Recall that homotopy equivalence between the topological spaces M, N means the existence of continuous maps $F : M \to N$ and $G : N \to M$ such that both $F \circ G$ and $G \circ F$ are homotopic (i.e. continuously deformable) to identity maps.

The answer is no, and this is known as the topological Poincaré conjecture, and is usually stated as follows: any homotopy n-sphere is homeomorphic to the n-sphere. It was proven for $n > 4$ by Smale, for $n = 4$ by Freedman, and for $n = 3$ is equivalent to the smooth Poincaré conjecture which was proved by Hamilton-Perelman. In dimensions $n = 1, 2$ it is a consequence of the classification of topological 1- and 2-manifolds.

Remark 1.7 (The Hausdorff and second countability axioms)**.** Without the Hausdorff assumption, we would have examples such as the following: take the disjoint union $R_1 \sqcup R_2$ of two copies of the real line, i.e. $R_1 = R_2 = \mathbb{R}$, and form the quotient by the equivalence relation generated by

$$R_1 \setminus \{0\} \ni x \sim \varphi(x) \in R_2 \setminus \{0\}, \tag{6}$$

where $\varphi : R_1 \setminus \{0\} \to R_2 \setminus \{0\}$ is defined by $\varphi(x) = x$. The resulting quotient topological space is locally homeomorphic to $\mathbb{R}$ but the points $[0 \in R_1], [0 \in R_2]$ cannot be separated by open neighbourhoods.

Second countability is a property which holds in most applications and is a necessary hypothesis in several useful theorems in the subject, as we shall see in the proof of the Whitney embedding theorem.

Example 1.8 (Projective spaces)**.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $\mathbb{K}P^n$ is defined to be the space of lines through $\{0\}$ in $\mathbb{K}^{n+1}$, and is called the projective space over $\mathbb{K}$ of dimension n.

More precisely, let $X = \mathbb{K}^{n+1} \backslash \{0\}$ and define an equivalence relation on X via $x \sim y$ iff $\exists \lambda \in \mathbb{K}^* = \mathbb{K} \backslash \{0\}$ such that $\lambda x = y$, i.e. x, y lie on the same line through the origin. The projective space is then the topological quotient

$$\mathbb{K}P^n = X/\sim .$$

The projection map $\pi : X \longrightarrow \mathbb{K}P^n$ is an *open* map, since if $U \subset X$ is open, then tU is also open $\forall t \in \mathbb{K}^*$, implying that $\cup_{t\in\mathbb{K}^*} tU = \pi^{-1}(\pi(U))$ is open, implying $\pi(U)$ is open. This immediately shows, by the way, that $\mathbb{K}P^n$ is second countable.

To show $\mathbb{K}P^n$ is Hausdorff (which we must do, since Hausdorff is preserved by subspaces and products, but *not* quotients), we show that the graph of the equivalence relation is closed in $X \times X$. Since π, and hence $\pi \times \pi$ are open, this implies that the diagonal is closed in $\mathbb{K}P^n \times \mathbb{K}P^n$, which is equivalent to the Hausdorff property. The graph in question is by definition

$$\Gamma_\sim = \{(x, y) \in X \times X \;:\; x \sim y\},$$

and we notice that $\Gamma_\sim$ is actually the common zero set of the following continuous functions

$$f_{ij}(x, y) = (x_i y_j - x_j y_i) \quad i \neq j,$$

implying at once that it is a closed subset.

An atlas for $\mathbb{K}P^n$ is given by the open sets $U_i = \pi(\tilde{U}_i)$, where

$$\tilde{U}_i = \{(x_0, \ldots, x_n) \in X \;:\; x_i \neq 0\},$$

and these are equipped with charts to $\mathbb{K}^n$ given by

$$\varphi_i([x_0, \ldots, x_n]) = x_i^{-1}(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \tag{7}$$

which are indeed invertible by $(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n)$.

Sometimes one finds it useful to simply use the "coordinates" $(x_0, \ldots, x_n)$ for $\mathbb{K}P^n$, with the understanding that the x_i are well-defined only up to overall rescaling. This is called using "projective coordinates" and in this case a point in $\mathbb{K}P^n$ is denoted by $[x_0 : \cdots : x_n]$.

Example 1.9 (Connected sum)**.** Let $p \in M$ and $q \in N$ be points in topological manifolds and let (U, φ) and (V, ψ) be charts around p, q such that $\varphi(p) = 0$ and $\psi(q) = 0$.

Choose ϵ small enough so that $B(0, 2\epsilon) \subset \varphi(U)$ and $B(0, 2\epsilon) \subset \varphi(V)$, and define the map of annuli

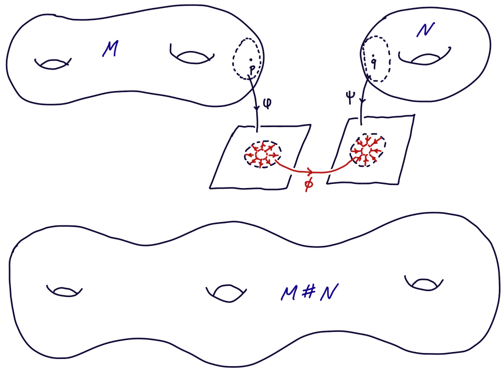
$$\begin{array}{ccc} B(0,2\epsilon)\backslash\overline{B(0,\epsilon)} & \xrightarrow{\phi} & B(0,2\epsilon)\backslash\overline{B(0,\epsilon)} \\ x & \longmapsto & \frac{2\epsilon^2}{|x|^2}x \end{array} \tag{8}$$

This is a homeomorphism of the annulus to itself, exchanging the boundaries. Now we define a new topological manifold, called the *connected sum* $M\#N$, as the quotient $X/\sim$, where

$$X = (M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}) \sqcup (N\backslash\overline{\psi^{-1}(B(0,\epsilon))}),$$

and we define an identification $x \sim \psi^{-1}\phi\varphi(x)$ for $x \in \varphi^{-1}(B(0, 2\epsilon))$. If $\mathcal{A}_M$ and $\mathcal{A}_N$ are atlases for M, N respectively, then a new atlas for the connect sum is simply

$$\mathcal{A}_M\big|_{M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}} \cup \mathcal{A}_N\big|_{N\backslash\overline{\psi^{-1}(B(0,\epsilon))}}.$$



Remark 1.10. The connected sum operation as described above may be viewed as an operation on the pair $(L, \{p, q\})$, where $L = M \sqcup N$ is the manifold formed by the disjoint union of M and N and $\{p, q\} \subset L$ is a set of two distinct points. The output of the connected sum is then the manifold $X/\sim$, where $\sim$ is as above and

$$X = L \backslash (\overline{\varphi^{-1}(B(0,\epsilon))} \sqcup \overline{\psi^{-1}(B(0,\epsilon))}).$$

The advantage of this formulation is that p, q need not be in the same connected component: indeed we may perform the connected sum of any manifold L with itself along a pair of points.

Remark 1.11. The homeomorphism type of the connected sum of connected manifolds M, N is independent of the choices of p, q and φ, ψ, except that it may depend on the two possible orientations of the gluing map $\psi^{-1}\phi\varphi$. To prove this, one must appeal to the so-called *annulus theorem*.

Remark 1.12. By iterated connect sum of S^2 with T^2 and $\mathbb{R}P^2$, we can obtain all compact 2-dimensional manifolds.

Example 1.13. Let F be a topological space. A fiber bundle with fiber F is a triple (E, p, B), where E, B are topological spaces called the "total space" and "base", respectively, and $p : E \longrightarrow B$ is a continuous surjective map called the "projection map", such that, for each point $b \in B$, there is a neighbourhood U of b and a homeomorphism

$$\Phi : p^{-1}U \longrightarrow U \times F,$$

such that $p_U \circ \Phi = p$, where $p_U : U \times F \longrightarrow U$ is the usual projection. The submanifold $p^{-1}(b) \cong F$ is called the "fiber over b".

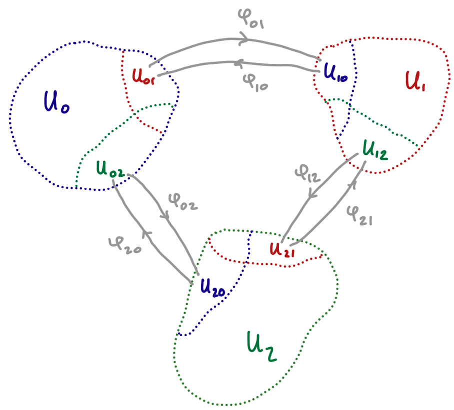
When B, F are topological manifolds, then clearly E becomes one as well. We will often encounter such manifolds.

Example 1.14 (General gluing construction)**.** To construct a topological manifold "from scratch", we glue open subsets of $\mathbb{R}^n$ together using homeomorphisms, as follows.

Begin with a countable collection of open subsets of $\mathbb{R}^n$: $\mathcal{A} = \{U_i\}$. Then for each i, we choose finitely many open subsets $U_{ij} \subset U_i$ and gluing maps

$$U_{ij} \xrightarrow{\varphi_{ij}} U_{ji} \ , \tag{9}$$

which we require to satisfy $\varphi_{ij}\varphi_{ji} = \mathrm{Id}_{U_{ji}}$, as well as $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all k, and most important of all, φ_{ij} must be *homeomorphisms*.



Next, we want the pairwise gluings to be consistent (transitive) and so we require that $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \mathrm{Id}_{U_{ij} \cap U_{jk}}$ for all i, j, k. This will ensure that the equivalence relation in (11) is well-defined.

Second countability of the glued manifold is guaranteed since we started with a countable collection of opens, but the Hausdorff property is not necessarily satisfied without a further assumption: we require that the graph of φ_{ij}, namely

$$\{(x, \varphi_{ij}(x)) \ : x \in U_{ij}\} \tag{10}$$

is a closed subset of $U_i \times U_j$.

The final glued topological manifold is then

$$M = \frac{\bigsqcup U_i}{\sim}, \tag{11}$$

for the equivalence relation generated by $x \sim \varphi_{ij}(x)$ for $x \in U_{ij}$, for all i, j. This space has a distinguished atlas $\mathcal{A}$, whose charts are simply the inclusions of the U_i in $\mathbb{R}^n$.

Example 1.15 (Quotient construction)**.** Let Γ be a group, and give it the discrete topology. Suppose Γ acts continuously on the topological n-manifold M, meaning that the action map

$$\begin{aligned} \Gamma \times M &\xrightarrow{\rho} M \\ (h, x) &\longmapsto h \cdot x \end{aligned}$$

is continuous. Suppose also that the action is *free*, i.e. the stabilizer of each point is trivial. Suppose the action is *properly discontinuous*, meaning that each $x \in M$ has a neighbourhood U such that $h \cdot U$ is disjoint from U for all $h \neq 1$. Finally, assume that the following subset is closed:

$$\{(x, y) \in M \times M \ : \ y = h \cdot x \text{ for some } h \in \Gamma\}$$

Then M/Γ is a topological manifold and $\pi : M \to M/\Gamma$ is a local homeomorphism.

Example 1.16 (Mapping torus)**.** Let M be a topological manifold and $\phi : M \to M$ a homeomorphism. Then

$$M_\phi = (M \times \mathbb{R})\,/\mathbb{Z}$$

is a manifold, where $k \in \mathbb{Z}$ acts via $k \cdot (p, t) = (\phi^k(p), t + k)$. This is called the mapping torus of ϕ and is a fibre bundle over $\mathbb{R}/\mathbb{Z} \cong S^1$ with fibre M.