1.2 Smooth manifolds

Given coordinate charts (U_i, φ_i) and (U_j, φ_j) on a topological manifold, we can compare them along the intersection $U_{ij} = U_i \cap U_j$, by forming the "gluing map"

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij}). \tag{12}$$

This is a homeomorphism, since it is a composition of homeomorphisms. In this sense, topological manifolds are glued together by homeomorphisms.

This means that a given function on the manifold may happen to be differentiable in one chart but not in another, if the gluing map between the charts is not smooth – there is no way to make sense of calculus on topological manifolds. This is why we introduce smooth manifolds, where the gluing maps are *smooth*.

Remark 1.17 (Aside on smooth maps of vector spaces)**.** Let $U \subset V$ be an open set in a finite-dimensional vector space, and let $f : U \longrightarrow W$ be a function with values in another vector space W. We say f is differentiable at $p \in U$ if there is a linear map $Df(p) : V \longrightarrow W$ which approximates f near p, meaning that

$$\lim_{\substack{x\to 0\\ x\neq 0}} \frac{|f(p+x) - f(p) - Df(p)(x)|}{|x|} = 0. \tag{13}$$

Notice that $Df(p)$ is uniquely characterized by the above property.

We have implicitly chosen inner products, and hence norms, on V and W in the above definition, though the differentiability of f is independent of this choice, since all norms are equivalent in finite dimensions. This is no longer true for infinite-dimensional vector spaces, where the norm or topology must be clearly specified and $Df(p)$ is required to be a continuous linear map. Most of what we do in this course can be developed in the setting of Banach spaces, i.e. complete normed vector spaces.

A basis for V has a corresponding dual basis $(x_1, \ldots, x_n)$ of linear functions on V, and we call these "coordinates". Similarly, let $(y_1, \ldots, y_m)$ be coordinates on W. Then the vector-valued function f has m scalar components $f_j = y_j \circ f$, and then the linear map $Df(p)$ may be written, relative to the chosen bases for V, W, as an $m \times n$ matrix, called the *Jacobian matrix* of f at p.

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{14}$$

We say that f is differentiable in U when it is differentiable at all $p \in U$, and we say it is continuously differentiable when

$$Df : U \longrightarrow \mathrm{Hom}(V, W) \tag{15}$$

is continuous. The vector space of continuously differentiable functions on U with values in W is called $C^1(U, W)$.

Notice that the first derivative Df is itself a map from U to a vector space $\mathrm{Hom}(V, W)$, so if its derivative exists, we obtain a map

$$D^2 f : U \longrightarrow \mathrm{Hom}(V, \mathrm{Hom}(V, W)), \tag{16}$$

and so on. The vector space of k times continuously differentiable functions on U with values in W is called $C^k(U, W)$. We are most interested in C^∞ or "smooth" maps, all of whose derivatives exist; the space of these is denoted $C^\infty(U, W)$, and so we have

$$C^\infty(U, W) = \bigcap_k C^k(U, W). \tag{17}$$

Note: for a C^2 function, $D^2 f$ actually has values in a smaller subspace of $V^* \otimes V^* \otimes W$, namely in $\mathrm{Sym}^2(V^*) \otimes W$, since "mixed partials are equal".

Definition 1.18. A *smooth manifold* is a topological manifold equipped with an equivalence class of smooth atlases, as explained next.

Definition 1.19. An atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ for a topological manifold is called *smooth* when all gluing maps

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij}) \tag{18}$$

are smooth maps, i.e. lie in $C^\infty(\varphi_i(U_{ij}), \mathbb{R}^n)$. Two atlases $\mathcal{A}, \mathcal{A}'$ are *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is itself a smooth atlas.

Remark 1.20. Note that the gluing maps $\varphi_j \circ \varphi_i^{-1}$ are not necessarily defined on all of $\mathbb{R}^n$. They only need be smooth on the open subset $\varphi_i(U_i \cap U_j) \subset \mathbb{R}^n$.

Remark 1.21. Instead of requiring an atlas to be smooth, we could ask for it to be C^k, or real-analytic, or even holomorphic (this makes sense for a $2n$-dimensional topological manifold when we identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$). This is how we define C^k, real-analytic, and complex manifolds, respectively.

We may now verify that all the examples from §1.1 are actually smooth manifolds:

Example 1.22 (Spheres)**.** The charts for the n-sphere given in Example 1.5 form a smooth atlas, since

$$\varphi_N \circ \varphi_S^{-1} : \vec{z} \mapsto \tfrac{1-x_0}{1+x_0} \vec{z} = \tfrac{(1-x_0)^2}{|\vec{x}|^2} \vec{z} = |\vec{z}|^{-2} \vec{z} \tag{19}$$

is a smooth map $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$, as required.

The Cartesian product of smooth manifolds inherits a natural smooth structure from taking the Cartesian product of smooth atlases. Hence the n-torus, for example, equipped with the atlas we described in Example 1.4, is smooth. Example 1.2 is clearly defining a smooth manifold, since the restriction of a smooth map to an open set is always smooth.

Example 1.23 (Projective spaces)**.** The charts for projective spaces given in Example 1.8 form a smooth atlas, since

$$\varphi_1 \circ \varphi_0^{-1}(z_1, \ldots, z_n) = (z_1^{-1}, z_1^{-1} z_2, \ldots, z_1^{-1} z_n), \tag{20}$$

which is smooth on $\mathbb{R}^n \setminus \{z_1 = 0\}$, as required, and similarly for all φ_i, φ_j.

The two remaining examples were constructed by gluing: the connected sum in Example 1.9 is clearly smooth since ϕ is a smooth map, and any topological manifold from Example 1.14 will be endowed with a natural smooth atlas as long as the gluing maps φ_{ij} are chosen to be C^∞.

1.3 Smooth maps

For topological manifolds M, N of dimension m, n, the natural notion of morphism from M to N is that of a continuous map. A continuous map with continuous inverse is then a homeomorphism from M to N, which is the natural notion of equivalence for topological manifolds. Since the composition of continuous maps is continuous, we obtain a "category" of topological manifolds and continuous maps.

A category is a collection of objects $\mathcal{C}$ (in our case, topological manifolds) and a collection of arrows $\mathcal{A}$ (in our case, continuous maps). Each arrow goes from an object (the source) to another object (the target), meaning that there are "source" and "target" maps from $\mathcal{A}$ to $\mathcal{C}$:

$$\mathcal{A} \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{C} \tag{21}$$

Also, a category has an identity arrow for each object, given by a map $\mathrm{id} : \mathcal{C} \longrightarrow \mathcal{A}$ (in our case, the identity map of any manifold to itself). Furthermore, there is an associative composition operation on arrows.

Conventionally, we write the set of arrows from X to X as $\mathrm{Hom}(X, Y)$, i.e.

$$\mathrm{Hom}(X, Y) = \{a \in \mathcal{A} \ : \ s(a) = X \text{ and } t(a) = Y\}. \tag{22}$$

Then the associative composition of arrows mentioned above becomes a map

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z). \tag{23}$$

We have described the category of topological manifolds; we now describe the category of smooth manifolds by defining the notion of a smooth map.

Definition 1.24. A continuous map $f : M \to N$ is called *smooth* when for each chart (U, φ) for M and each chart (V, ψ) for N, the composition $\psi \circ f \circ \varphi^{-1}$ is a smooth map where it is defined, i.e. from the open set $\varphi(f^{-1}(V))$ to $\mathbb{R}^n$:

The set of smooth maps (i.e. morphisms) from M to N is denoted $C^\infty(M, N)$. A smooth map with a smooth inverse is called a *diffeomorphism*.

Proposition 1.25. *If $g : L \to M$ and $f : M \to N$ are smooth maps, then so is the composition $f \circ g$.*

Proof. If charts φ, χ, ψ for L, M, N are chosen near $p \in L$, $g(p) \in M$, and $(fg)(p) \in N$, then $\psi \circ (f \circ g) \circ \varphi^{-1} = A \circ B$, for $A = \psi f \chi^{-1}$ and $B = \chi g \varphi^{-1}$ both smooth mappings $\mathbb{R}^n \to \mathbb{R}^n$. By the chain rule, $A \circ B$ is differentiable at p, with derivative $D_{\phi(p)}(A \circ B) = (D_{\chi(g(p))}A)(D_{\phi(p)}B)$ (matrix multiplication). $\square$

Now we have a new category, the category of smooth manifolds and smooth maps; two manifolds are considered isomorphic when they are diffeomorphic.

Example 1.26. The smooth inclusion $j : S^1 \to \mathbb{C}$ induces a smooth inclusion $j \times j$ of the 2-torus $T^2 = S^1 \times S^1$ into $\mathbb{C}^2$. The image of $j \times j$ does not include zero, so we may compose with the projection $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$ and the diffeomorphism $\mathbb{C}P^1 \to S^2$, to obtain a smooth map

$$\pi \circ (j \times j) : T^2 \to S^2. \tag{24}$$

Remark 1.27 (Exotic smooth structures)**.** The topological Poincaré conjecture, now proven, states that any topological manifold homotopic to the n-sphere is in fact homeomorphic to it. We have now seen how to put a differentiable structure on this n-sphere. Remarkably, there are other differentiable structures on the n-sphere which are not diffeomorphic to the standard one we gave; these are called *exotic* spheres.

Since the connected sum of spheres is homeomorphic to a sphere, and since the connected sum operation is well-defined as a smooth manifold, it follows that the connected sum defines a *monoid* structure on the set of smooth n-spheres. In fact, Kervaire and Milnor showed that for $n \neq 4$, the set of (oriented) diffeomorphism classes of smooth n-spheres forms a finite abelian group under the connected sum operation. This is not known to be the case in four dimensions. Kervaire and Milnor also compute the order of this group, and the first dimension where there is more than one smooth sphere is $n = 7$, in which case they show there are 28 smooth spheres, which we will encounter later on.

The situation for spheres may be contrasted with that for the Euclidean spaces: any differentiable manifold homeomorphic to $\mathbb{R}^n$ for $n \neq 4$ must be diffeomorphic to it. On the other hand, by results of Donaldson, Freedman, Taubes, and Kirby, we know that there are uncountably many non-diffeomorphic smooth structures on the topological manifold $\mathbb{R}^4$; these are called *fake* $\mathbb{R}^4$s.

Remark 1.28. The maps $\alpha : x \mapsto x$ and $\beta : x \mapsto x^3$ are both homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$. Each one defines, by itself, a smooth atlas on $\mathbb{R}$. These two smooth atlases are not compatible (why?), so they do not define the same smooth structure on $\mathbb{R}$. Nevertheless, the smooth structures are equivalent, since there is a diffeomorphism taking one to the other. What is it?

Example 1.29 (Lie groups)**.** A group is a set G with an associative multiplication $G \times G \xrightarrow{m} G$, an identity element $e \in G$, and an inversion map $\iota : G \longrightarrow G$, usually written $\iota(g) = g^{-1}$.

If we endow G with a topology for which G is a topological manifold and m, ι are continuous maps, then the resulting structure is called a *topological group*. If G is a given a smooth structure and m, ι are smooth maps, the result is a *Lie group*.

The real line (where m is given by addition), the circle (where m is given by complex multiplication), and their Cartesian products give simple but important examples of Lie groups. We have also seen the general linear group $GL(n, \mathbb{R})$, which is a Lie group since matrix multiplication and inversion are smooth maps.

Since $m : G \times G \longrightarrow G$ is a smooth map, we may fix $g \in G$ and define smooth maps $L_g : G \longrightarrow G$ and $R_g : G \longrightarrow G$ via $L_g(h) = gh$ and $R_g(h) = hg$. These are called *left multiplication* and *right multiplication*. Note that the group axioms imply that $R_g L_h = L_h R_g$.