2 The derivative

The derivative of a smooth map is an absolutely central concept in differential geometry. To make sense of the derivative, however, we must introduce the notion of tangent vector and, further, the space of all tangent vectors, known as the tangent bundle. In this section, we describe the tangent bundle intrinsically, without reference to any embedding of the manifold in a vector space. The definition of the tangent bundle is simplest for an open subset $U \subset \mathbb{R}^n$. In this case, a tangent vector to a point $p \in U$ is simply a vector in $\mathbb{R}^n$, and so the tangent bundle, which consists of all tangent vectors to all points in U, is simply given by

$$TU = U \times \mathbb{R}^n. \tag{25}$$

We now investigate the problem of generalizing the tangent bundle to other manifolds, where the convenience of being an open set in a vector space is not available.

2.1 The tangent bundle

The tangent bundle of an n-manifold M is a $2n$-manifold, called TM, naturally constructed in terms of M. As a set, it is fairly easy to describe, as simply the disjoint union of all tangent spaces. However we must explain precisely what we mean by the tangent space T_pM to $p \in M$.

We may define a tangent vector v is as an equivalence class of smooth curves. Let a smooth curve through p be a smooth map $\gamma : I \to M$ from an open interval around zero $I \subset \mathbb{R}$ to the manifold M, such that $\gamma(0) = p$. Then we say two such curves γ_1, γ_2 are equivalent when they have the same *velocity* at p, which we take to mean the following: in a chart (U, φ) containing p, we have

$$\tfrac{d}{dt}\Big|_{t=0}(\varphi \circ \gamma_1) = \tfrac{d}{dt}\Big|_{t=0}(\varphi \circ \gamma_2).$$

Note that the above differentiation makes sense since $\varphi \circ \gamma_i$ are maps between Euclidean spaces, which we know how to differentiate. Also note that if this condition holds in one chart, then it clearly holds in any other chart, by the chain rule.

Inspired by the above definition, which uses charts to make sense of the derivative of a curve, we now present an alternative definition which emphasizes the importance of the charts and makes it more clear how tangent spaces at different points may be unified to obtain a single tangent bundle. We use as main ingredient the definition (25) of the tangent bundle of an open set in Euclidean space.

Definition 2.1. Let $(U, \varphi), (V, \psi)$ be coordinate charts around $p \in M$. Let $u \in T_{\varphi(p)}\varphi(U)$ and $v \in T_{\psi(p)}\psi(V)$. Then the triples (U, φ, u), (V, ψ, v) are called equivalent when $D(\psi \circ \varphi^{-1})(\varphi(p)) : u \mapsto v$. The chain rule for derivatives $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ guarantees that this is indeed an equivalence relation.

The set of equivalence classes of such triples is called the tangent space to p of M, denoted T_pM. It is a real vector space of dimension $\dim M$, since both $T_{\varphi(p)}\varphi(U)$ and $T_{\psi(p)}\psi(V)$ are, and $D(\psi \circ \varphi^{-1})$ is a linear isomorphism.

As a set, the tangent bundle is defined by

$$TM = \bigsqcup_{p \in M} T_pM, \tag{26}$$

and it is equipped with a natural surjective map $\pi : TM \longrightarrow M$, which is simply $\pi(X) = x$ for $X \in T_xM$.

We now give it a manifold structure in a natural way.

Proposition 2.2. *For an n-manifold M, the set TM has a natural topology and smooth structure which make it a $2n$-manifold, and make $\pi : TM \longrightarrow M$ a smooth map.*

Proof. Any chart (U, φ) for M defines a bijection

$$T\varphi(U) \cong U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U) \tag{27}$$

via $(p, v) \mapsto (U, \varphi, v)$. Using this, we induce a smooth manifold structure on $\pi^{-1}(U)$, and view the inverse of this map as a chart $(\pi^{-1}(U), \Phi)$ to $\varphi(U) \times \mathbb{R}^n$.

given another chart (V, ψ), we obtain another chart $(\pi^{-1}(V), \Psi)$ and we may compare them via

$$\Psi \circ \Phi^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n, \tag{28}$$

which is given by $(p, u) \mapsto ((\psi \circ \varphi^{-1})(p), D(\psi \circ \varphi^{-1})_p u)$, which is smooth. Therefore we obtain a topology and smooth structure on all of TM (by defining W to be open when $W \cap \pi^{-1}(U)$ is open for every U in an atlas for M; all that remains is to verify the Hausdorff property, which holds since points x, y are either in the same chart (in which case it is obvious) or they can be separated by the given type of charts. $\square$

Remark 2.3. This is a more constructive way of looking at the tangent bundle: We choose a countable, locally finite atlas $\{(U_i, \varphi_i)\}$ for M and glue together $U_i \times \mathbb{R}^n$ to $U_j \times \mathbb{R}^n$ via an equivalence

$$(x, u) \sim (y, v) \ \Leftrightarrow\ y = \varphi_j \circ \varphi_i^{-1}(x) \ \text{ and } \ v = D(\varphi_j \circ \varphi_i^{-1})_x u, \tag{29}$$

and verify the conditions of the general gluing construction 1.14. The choice of a different atlas yields a canonically diffeomorphic manifold.

2.2 The derivative

A description of the tangent bundle is not complete without defining the derivative of a general smooth map of manifolds $f : M \longrightarrow N$. Such a map may be defined locally in charts (U_i, φ_i) for M and (V_α, ψ_α) for N as a collection of vector-valued functions $\psi_\alpha \circ f \circ \varphi_i^{-1} = f_{i\alpha}$ (defined where the composition makes sense) which satisfy (again, at all points where the composition is defined)

$$(\psi_\beta \circ \psi_\alpha^{-1}) \circ f_{i\alpha} = f_{j\beta} \circ (\varphi_j \circ \varphi_i^{-1}). \tag{30}$$

Differentiating, we obtain

$$D(\psi_\beta \circ \psi_\alpha^{-1}) \circ Df_{i\alpha} = Df_{j\beta} \circ D(\varphi_j \circ \varphi_i^{-1}). \tag{31}$$

Equation 31 shows that $Df_{i\alpha}$ and $Df_{j\beta}$ glue together to define a map $TM \longrightarrow TN$. This map is called the derivative of f and is denoted $Df : TM \longrightarrow TN$. Sometimes it is called the "push-forward" of vectors and is denoted f_*. The map fits into the commutative diagram

$$\begin{tikzcd} TM \arrow[r, "Df"] \arrow[d, "\pi"'] & TN \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & N \end{tikzcd} \tag{32}$$

Each fiber $\pi^{-1}(x) = T_xM \subset TM$ is a vector space, and the map $Df : T_xM \longrightarrow T_{f(x)}N$ is a linear map. In fact, (f, Df) defines a homomorphism of vector bundles from TM to TN.

The usual chain rule for derivatives then implies that if $f \circ g = h$ as maps of manifolds, then $Df \circ Dg = Dh$. As a result, we obtain the following category-theoretic statement.

Proposition 2.4. *The mapping T which assigns to a manifold M its tangent bundle TM, and which assigns to a map $f : M \longrightarrow N$ its derivative $Df : TM \longrightarrow TN$, is a functor from the category of manifolds and smooth maps to itself*[1].

For this reason, the derivative map Df is sometimes called the "tangent mapping" Tf.

2.3 Local structure of smooth maps

In some ways, smooth manifolds are easier to produce or find than general topological manifolds, because of the fact that smooth maps have linear approximations. Therefore smooth maps often behave like linear maps of vector spaces, and we may gain inspiration from vector space constructions (e.g. subspace, kernel, image, cokernel) to produce new examples of manifolds.

In charts (U, φ), (V, ψ) for the smooth manifolds M, N, a smooth map $f : M \longrightarrow N$ is represented by a smooth map $\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$. We shall give a general local classification of such maps, based on the behaviour of the derivative. The fundamental result which provides information about the map based on its derivative is the *inverse function theorem.*

Theorem 2.5 (Inverse function theorem)**.** *Let $f : (M, p) \to (N, q)$ be a smooth map of n-dimensional manifolds and suppose that $Df(p) : T_pM \to T_qN$ is invertible. Then f has a local smooth inverse. That is, there are neighbourhoods U, V of p, q and a smooth map $g : V \to U$ such that $f \circ g = \mathrm{id}_V$ and $g \circ f = \mathrm{id}_U$.*

This theorem provides us with a local normal form for a smooth map with $Df(p)$ invertible: we may choose coordinates on sufficiently small

[1] We can also say that it is a functor from manifolds to the category of smooth vector bundles.

neighbourhoods of $p, f(p)$ so that f is represented by the identity map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

In fact, the inverse function theorem leads to a normal form theorem for a more general class of maps:

Theorem 2.6 (Constant rank theorem)**.** *Let $f : M^m \to N^n$ be a smooth map such that Df has constant rank k in a neighbourhood of $p \in M$. Then there are charts (U, φ) and (V, ψ) containing $p, f(p)$ such that*

$$\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0). \tag{33}$$

Proof. Begin by choosing charts so that without loss of generality M is an open set in $\mathbb{R}^m$ and N is $\mathbb{R}^n$.

Since rk $Df = k$ at p, there is a $k \times k$ minor of $Df(p)$ with nonzero determinant. Reorder the coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$ so that this minor is top left, and translate coordinates so that $f(0) = 0$. label the coordinates $(x_1, \ldots, x_k, y_1, \ldots y_{m-k})$ on the domain and $(u_1, \ldots u_k, v_1, \ldots, v_{n-k})$ on the codomain.

Then we may write $f(x, y) = (Q(x, y), R(x, y))$, where Q is the projection to $u = (u_1, \ldots, u_k)$ and R is the projection to v. with $\frac{\partial Q}{\partial x}$ nonsingular. First we wish to put Q into normal form. Consider the map $\phi(x, y) = (Q(x, y), y)$, which has derivative

$$D\phi = \begin{pmatrix} \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \\ 0 & 1 \end{pmatrix} \tag{34}$$

As a result we see $D\phi(0)$ is nonsingular and hence there exists a local inverse $\phi^{-1}(x, y) = (A(x, y), B(x, y))$. Since it's an inverse this means $(x, y) = \phi(\phi^{-1}(x, y)) = (Q(A, B), B)$, which implies that $B(x, y) = y$.

Then $f \circ \phi^{-1} : (x, y) \mapsto (x, S = R(A, y))$, and must still be of rank k. Since its derivative is

$$D(f \circ \phi^{-1}) = \begin{pmatrix} I_{k\times k} & 0 \\ \frac{\partial S}{\partial x} & \frac{\partial S}{\partial y} \end{pmatrix} \tag{35}$$

we conclude that $\frac{\partial S}{\partial y} = 0$, meaning that we have eliminated the y-dependence:

$$f \circ \phi^{-1} : (x, y) \mapsto (x, S(x)). \tag{36}$$

We now postcompose by the diffeomorphism $\sigma : (u, v) \mapsto (u, v - S(u))$, to obtain

$$\sigma \circ f \circ \phi^{-1} : (x, y) \mapsto (x, 0), \tag{37}$$

as required. □

As we shall see, these theorems have many uses. One of the most straightforward uses is for defining submanifolds.

There are several ways to define the notion of submanifold. We will use a definition which works for topological and smooth manifolds, based on the local model of inclusion of a vector subspace. These are sometimes called *regular* or *embedded* submanifolds.

Definition 2.7. A subspace $L \subset M$ of an m-manifold is called a submanifold of codimension k when each point $x \in L$ is contained in a chart (U, φ) for M such that

$$L \cap U = f^{-1}(0), \tag{38}$$

where f is the composition of φ with the projection $\mathbb{R}^m \to \mathbb{R}^k$ to the last k coordinates $(x_{m-k+1}, \ldots, x_m)$. A submanifold of codimension 1 is usually called a *hypersurface*.

Proposition 2.8. *If $f : M \longrightarrow N$ is a smooth map of manifolds, and if $Df(p)$ has constant rank on M, then for any $q \in f(M)$, the inverse image $f^{-1}(q) \subset M$ is a regular submanifold.*

Proof. Let $x \in f^{-1}(q)$. Then there exist charts ψ, φ such that $\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$ and $f^{-1}(q) \cap U = \{x_1 = \cdots = x_k = 0\}$. Hence we obtain that $f^{-1}(q)$ is a codimension k submanifold. □

Example 2.9. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by $(x_1, \ldots, x_n) \mapsto \sum x_i^2$. Then $Df(x) = (2x_1, \ldots, 2x_n)$, which has rank 1 at all points in $\mathbb{R}^n \backslash \{0\}$. Hence since $f^{-1}(q)$ contains $\{0\}$ iff $q = 0$, we see that $f^{-1}(q)$ is a regular submanifold for all $q \neq 0$. Exercise: show that this manifold structure is compatible with that obtained in Example 1.22.

The previous example leads to the following special case.

Proposition 2.10. *If $f : M \longrightarrow N$ is a smooth map of manifolds and $Df(p)$ has rank equal to* $\dim N$ *along $f^{-1}(q)$, then this subset $f^{-1}(q)$ is an embedded submanifold of M.*

Proof. Since the rank is maximal along $f^{-1}(q)$, it must be maximal in an open neighbourhood $U \subset M$ containing $f^{-1}(q)$, and hence $f : U \longrightarrow N$ is of constant rank. □

Definition 2.11. If $f : M \longrightarrow N$ is a smooth map such that $Df(p)$ is surjective, then p is called a *regular point*. Otherwise p is called a *critical point*. If all points in the level set $f^{-1}(q)$ are regular points, then q is called a *regular value*, otherwise q is called a critical value. In particular, if $f^{-1}(q) = \varnothing$, then q is regular.

It is often useful to highlight two classes of smooth maps; those for which Df is everywhere *injective*, or, on the other hand *surjective*.

Definition 2.12. A smooth map $f : M \longrightarrow N$ is called a *submersion* when $Df(p)$ is surjective at all points $p \in M$, and is called an *immersion* when $Df(p)$ is injective at all points $p \in M$. If f is an injective immersion which is a homeomorphism onto its image (when the image is equipped with subspace topology), then we call f an *embedding*.

Proposition 2.13. *If $f : M \longrightarrow N$ is an embedding, then $f(M)$ is a regular submanifold.*

Proof. Let $f : M \longrightarrow N$ be an embedding. Then for all $m \in M$, we have charts (U, φ), (V, ψ) where $\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$. If $f(U) = f(M) \cap V$, we're done. To make sure that some other piece of M doesn't get sent into the neighbourhood, use the fact that $f(U)$ is open in the subspace topology. This means we can find a smaller open set $V' \subset V$

such that $V' \cap f(M) = f(U)$. Restricting the coordinates to V', we see that $f(M)$ is cut out by $(x_{m+1}, \ldots, x_n)$, where $n = \dim N$. $\square$

Example 2.14. If $\iota : M \longrightarrow N$ is an embedding of M into N, then $D\iota : TM \longrightarrow TN$ is also an embedding (hence so are $D^k\iota : T^kM \longrightarrow T^kN$), showing that TM is a submanifold of TN.