Reading: Woit Chapter 2, 3, and 4. Also ensure that you get acquainted with the matrix exponential for $n \times n$ matrices, as well as the idea of tensor product of finite-dimensional vector spaces.

Notation: Throughout this assignment, V is the Hermitian vector space $\mathbb{C}^n$ with the Hermitian inner product

$$\langle u, v \rangle = \bar{u}^\top v = \sum_{i=1}^{n} \bar{u}_i v_i.$$

Exercise 1 (For your own review, not to be handed in)**.**

1. Let S, T be two linear operators $V \to V$ which *commute*, meaning $ST = TS$. Prove that S stabilizes every eigenspace of T, and prove that if both S and T are diagonalizable, then they are *simultaneously diagonalizable*, i.e., they are both diagonal with respect to a single choice of basis.

2. Let T be a linear operator as above and let $W \subset V$ be T-stable linear subspace. Show that the orthogonal complement $W^\perp$ is T^*-stable. Recall that T^* is the operator defined by the condition
$$\langle T^* v, w \rangle = \langle v, Tw \rangle.$$

3. An operator T is called *normal* when T commutes with T^*. Prove that a normal operator may be diagonalized by an appropriate choice of orthonormal basis.

4. Show that any Hermitian matrix can be diagonalized by a unitary transformation and that the resulting diagonal entries are real.

Exercise 2. Recall that $U(n)$ is the set of $n \times n$ matrices A satisfying

$$\langle Ax, Ay \rangle = \langle x, y \rangle \text{ for all } x, y, \text{ or equivalently } A^* A = \mathbf{1}.$$

1. Show that any unitary matrix can be diagonalized using a unitary change of basis. What are the possible diagonal entries in the result?

2. Show that any unitary matrix may be written as e^X for X a skew-adjoint matrix. Recall that the matrix exponential is given by the convergent power series
$$e^X = \sum_0^\infty \frac{1}{k!} X^k.$$

Exercise 3. Consider the Pauli matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which form a basis for the real vector space of self-adjoint operators on $\mathbb{C}^2$.

1. Compute the matrix exponential $e^{-itX}, t \in \mathbb{R}$, for X being each of the Pauli matrices.
2. Define the Hamiltonian operator $H = -B\sigma_1$ for a fixed $B \in \mathbb{R}$ and use it to evolve the state

$$\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

via Schrödinger unitary evolution. Describe the resulting path $\psi(t)$ in state space.

3. Compute the vector $J(t) = (\langle\sigma_1\rangle, \langle\sigma_2\rangle, \langle\sigma_3\rangle)$ of expectation values for the family of states $\psi(t)$. Compare and contrast the evolution of the real 3-vector $J(t)$ with the evolution of the complex 2-vector $\psi(t)$.

Exercise 4. Let (e_1, e_2) and (f_1, f_2) be bases for the vector spaces U, V respectively and let A, B be endomorphisms of U, V respectively, so that $Ae_i = \sum_j a_{ij} e_i$ and $Bf_i = \sum_j b_{ij} f_i$, expressing A and B as matrices $[a_{ij}]$ and $[b_{ij}]$ in the usual way. Ordering the basis of $U \otimes V$ lexicographically, namely $(e_1 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_1, e_2 \otimes f_2)$, write the matrix of the endomorphism $A \otimes B$.

Exercise 5. Let $e \in V$ be a fixed unit vector. Does there exist a unitary operator on $V \otimes V$ which takes $v \otimes e$ to $v \otimes v$ for all unit vectors v? Give an example/proof.

Exercise 6. Let E be a self-adjoint operator on V such that $E\,|i\rangle = i\,|i\rangle$ for $i = 0, 1$ and let $|ij\rangle$ denote the tensor product $|i\rangle \otimes |j\rangle$. Prove that the state

$$\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

in $V \otimes V$ is not decomposable. Exactly when will a general state

$$p\,|00\rangle + q\,|01\rangle + r\,|10\rangle + s\,|11\rangle$$

be entangled, i.e., non-decomposable?