**Reading:** Woit Chapter 9, 10, and a brief introduction to Hilbert spaces, I recommend: https://www.math.ucdavis.edu/~hunter/book/ch6.pdf

**Exercise 1.** Let  $V \cong \mathbb{C}^2$  be the standard spin- $\frac{1}{2}$  representation of SU(2), with standard basis vectors  $(e_0, e_1)$ . The representation  $S^2V \otimes S^2V$  decomposes into three irreducible representations; determine which ones. Give an argument using highest weight vectors. Then give explicit bases spanning the three summands, with each basis element expressed in terms of the elements

$$\ell_1 = \frac{i}{\sqrt{2}} (e_0 \otimes e_0 - e_1 \otimes e_1), \quad \ell_2 = \frac{1}{\sqrt{2}} (e_0 \otimes e_0 + e_1 \otimes e_1), \quad \ell_3 = \frac{-i}{\sqrt{2}} (e_0 \otimes e_1 + e_1 \otimes e_0).$$

Recall that this basis for  $S^2V$  may be identified with the basis  $(X_k = -\frac{i}{2}\sigma_k)_{k=1,2,3}$  for  $\mathfrak{su}(2)$ , defining an isomorphism between  $S^2V$  and the (complexified) adjoint representation.

**Exercise 2.** The Casimir operator in a representation  $\pi$  of  $\mathfrak{su}(2)$  on V is given by the operator (the superscript on L does not mean that it is a square).

$$L_{\pi}^{2} = (\pi(X_{1}))^{2} + (\pi(X_{2}))^{2} + (\pi(X_{3}))^{2},$$

where  $X_k = -\frac{i}{2}\sigma_k$  as before.

- 1. Prove that if we change the basis  $(X_1, X_2, X_3)$  for  $\mathfrak{su}(2)$  using conjugation by a matrix in SU(2), then the operator  $L^2_{\pi}$  remains unchanged. Hint: Show that  $L^2_{\pi}$  remains constant if we apply 1-parameter subgroups  $e^{tX_k}$  to the basis.
- 2. Compute the Casimir operator for the action of SU(2) on polynomial functions of  $(z_1, z_2) \in \mathbb{C}^2$ ; verify that the homogeneous polynomials are eigenvectors and determine their eigenvalues.
- 3. Let  $L^2_{\pi}$  and  $L^2_{\rho}$  be the Casimir operators for two SU(2) representations  $\pi, \rho$  on vector spaces U, V, and let  $L^2_{\pi \otimes \rho}$  be the Casimir for the tensor product representation on  $U \otimes V$ . Describe explicitly the operator  $\frac{1}{2}(L^2_{\pi \otimes \rho} L^2_{\pi} L^2_{\rho})$  using the basis  $(X_1, X_2, X_3)$ . This is the famous Spin-Orbit coupling operator.

**Exercise 3.** (Vector subspaces of a Hilbert space  $\mathcal{H}$ ) Let  $\ell^2$  be the Hilbert space of squaresummable sequences of complex numbers.

- 1. Give examples of subspaces of  $\ell^2$  which a) have infinite dimension and codimension, b) which have finite dimension, and c) which have finite codimension.
- 2. Give an example of a proper subspace of  $\ell^2$  which is closed.
- 3. Give an example of a proper subspace of  $\ell^2$  which is dense.
- 4. if  $W \subset \mathcal{H}$  is a subspace, show the closure  $\overline{W}$  is a subspace, show

$$W^{\perp} = (\overline{W})^{\perp},$$

and show that  $W \cap W^{\perp} = \{0\}.$ 

- 5. Show that if  $W \subset \mathcal{H}$  is a closed subspace, then W and  $\mathcal{H}/W$  naturally inherit a Hilbert space structure.
- 6. Is it possible that  $\overline{W}/W$  be nonzero but finite-dimensional?

## **Exercise 4.** (The unit sphere in Hilbert space). Let $S(\mathcal{H}) \subset \mathcal{H}$ be the unit sphere in $\mathcal{H}$ .

- 1. Show that  $S(\mathcal{H})$  is closed.
- 2. Show that a linear map of Hilbert spaces  $F : \mathcal{H}_1 \to \mathcal{H}_2$  is continuous if and only if  $F(S(\mathcal{H}_1))$  is bounded. (This is why such maps are sometimes called "bounded operators") Show this is equivalent to the inequality

$$||Fv||_{\mathcal{H}_2} \le C||v||_{\mathcal{H}_1} \quad \forall v \in \mathcal{H}_1, \tag{1}$$

for some constant C independent of v.

- 3. Suppose that  $D \subset \mathcal{H}_1$  is a dense linear subspace and  $F : D \to \mathcal{H}_2$  is a continuous linear map. Show that F has a unique extension to a continuous linear map  $F : \mathcal{H}_1 \to \mathcal{H}_2$ .
- 4. The operator  $\frac{d}{dx}$  is defined on a dense subspace  $D \subset L^2(\mathbb{R})$  containing the smooth functions with compact support (meaning that the function vanishes outside some finite interval in  $\mathbb{R}$ ). Show that  $\frac{d}{dx}$  is not bounded, that is, that  $\frac{d}{dx} : D \to L^2(\mathbb{R})$  is not continuous. Despite the fact that  $\frac{d}{dx}$  is not defined on all of  $L^2(\mathbb{R})$ , we refer to it as an operator on  $L^2(\mathbb{R})$ , keeping in mind that it is defined only on a dense subspace called its *domain*.

**Exercise 5.** (The continuous dual) The operator norm of a continuous linear map  $F : \mathcal{H}_1 \to \mathcal{H}_2$  is defined as

$$||F|| := \sup_{v \in S(\mathcal{H}_1)} ||Fv||_{\mathcal{H}_2}.$$

Show that the composition of continuous linear operators is a continuous operation in the operator norm, i.e. for A, B continuous linear operators, show

$$||A \circ B|| \le ||A||||B||.$$

Let  $\mathcal{H}'$  denote the continuous dual of  $\mathcal{H}$ , i.e. the space of continuous linear maps  $L : \mathcal{H} \to \mathbb{R}$ , equipped with operator norm, viewing  $\mathbb{R}$  as a Hilbert space.

1. Show that the "dualization map"  $v \mapsto v^* = \langle v, \cdot \rangle$  is an injective, norm-preserving continuous linear map  $\mathcal{H} \to \mathcal{H}'$ .

The Riesz representation theorem states that the dualization map is an isomorphism of Hilbert spaces.

2. Show that if  $F: \mathcal{H}_1 \to \mathcal{H}_2$  is a continous linear operator, then  $F^*: \mathcal{H}'_2 \to \mathcal{H}'_1$  defined by

$$F^*\mu = \mu \circ F$$

is a continuous linear map. If F is injective, under what conditions is  $F^*$  surjective? Show that if F is injective and Im(F) is dense, then  $F^*$  is injective.