Reading: Woit Chapter 22

Exercise 1. Recall that the 1-dimensional harmonic oscillator is the quantum system where $\mathcal{H} = L^2(\mathbb{R})$ and $H = \frac{1}{2}(P^2 + Q^2)$. We saw that H has a 1-dimensional space of ground states (lowest-eigenvalue states) generated by the eigenvector

$$|0\rangle = \pi^{-1/4} e^{-q^2/2}$$

with eigenvalue 1/2, and that by applying the raising operator $a^* = 2^{-1/2}(Q - iP)$ we produce a sequence of states

$$|n+1\rangle = (n+1)^{-1/2} a^* |n\rangle, n = 0, 1, 2, \dots$$

such that $|n\rangle$ has eigenvalue $E_n = n + 1/2$.

Prove that any eigenstate of H must lie in one of the eigenspaces listed above.

Exercise 2. Consider the 3-d harmonic oscillator, with $\mathcal{H} = L^2(\mathbb{R}^3)$ and Hamiltonian

$$H = \tfrac{1}{2}(P_1^2 + P_2^2 + P_3^2) + \tfrac{1}{2}(Q_1^2 + Q_2^2 + Q_3^2).$$

1. What are the eigenvalues of H?
2. Describe an explicit basis of eigenvectors for the eigenspaces corresponding to the three lowest eigenvalues.
3. The rotation group $SO(3)$ acts on $L^2(\mathbb{R}^3)$ and the Lie algebra generators l_1, l_2, l_3 are sent by this representation to

   $$\pi'(l_1) = -(q_2 \frac{\partial}{\partial q_3} - q_3 \frac{\partial}{\partial q_2}) \qquad \pi'(l_2) = -(q_3 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial q_3}) \qquad \pi'(l_3) = -(q_1 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial q_1})$$

   As a result, we may express the corresponding self-adjoint operators, known as the angular momentum operators, $L_i = i\pi'(l_i)$ in terms of the linear momenta and position operators, that is,

   $$L_1 = Q_2 P_3 - Q_3 P_2 \qquad L_2 = Q_3 P_1 - Q_1 P_3 \qquad L_3 = Q_1 P_2 - Q_2 P_1$$

   Determine whether the angular momentum observables are conserved in this system.
4. What does the previous result imply about the action of $SO(3)$ on the states found in question 2. Which irreducible representations occur?

Exercise 3. The perturbative series $\hat{F} \in \mathbb{C}[[\epsilon]]$ which describes the function

$$F(\epsilon) = \frac{\int \exp(-\frac{1}{2}ax^2 + \epsilon x^3/3!)}{\int \exp(-\frac{1}{2}ax^2)}$$

may be described as follows:

$$\hat{F}(\epsilon) = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} a^{-e} \epsilon^v,$$

where the sum is over all possible trivalent graphs Γ with e edges and v vertices, and $\mathrm{Aut}(\Gamma)$ is the symmetry group of the graph. Also, the empty graph is assigned 1.

To be precise: divide each edge of the graph into two half-edges. Then the graph may be viewed (in fact is defined) as a collection of half-edges together with two partitions: the partition into sets of size 2 called edges, and the partition into sets of size 3 called vertices. A symmetry of the graph is a bijection from the half-edges to themselves which preserves the two partitions.

Note that there are no trivalent graphs with only 1 vertex. So the power series must begin with $1 + c\epsilon^2 + \cdots$ for some constant c. We saw in class that there are two trivalent graphs with two vertices: the theta-graph (shaped like the letter θ) and the dumbbell graph (shaped like O—O). These graphs have symmetry group of size 12 (permute 3 edges or 2 vertices) and 8 (flip each of the 3 edges independently), respectively. So we see that up to the ϵ^2 term we have

$$\hat{F}(\epsilon) = 1 + \tfrac{1}{12}a^{-3}\epsilon^2 + \tfrac{1}{8}a^{-3}\epsilon^2 + \cdots .$$

Compute $\hat{F}(\epsilon)$ to the next order in perturbation theory – the coefficient of ϵ^3 is zero, so determine the coefficient of ϵ^4. In particular, you must find all trivalent graphs with exactly four vertices and study their automorphisms.

Mega Bonus: make a numerical comparison between $F(\epsilon)$ and its asymptotic expansion (for $a = 1$, say): you can define $F(\epsilon)$ (for ϵ chosen generically, not necessarily on the real axis) by integrating along the path of steepest descent of $\mathrm{Re}(-x^2/2 + \epsilon x^3/3!)$, or in other words along the path given by $\mathrm{Im}(-x^2/2 + \epsilon x^3/3!) = 0$.