# 1 Manifolds

A manifold is a space which looks like $\mathbb{R}^n$ at small scales (i.e. "locally"), but which may be very different from this at large scales (i.e. "globally"). In other words, manifolds are made by gluing pieces of $\mathbb{R}^n$ together to make a more complicated whole. We want to make this precise.

## 1.1 Topological manifolds

**Definition 1.1.** A real, n-dimensional *topological manifold* is a Hausdorff, second countable topological space which is locally homeomorphic to $\mathbb{R}^n$.

"Locally homeomorphic to $\mathbb{R}^n$" simply means that each point $p$ has an open neighbourhood $U$ for which we can find a homeomorphism $\varphi : U \longrightarrow V$ to an open subset $V \in \mathbb{R}^n$. Such a homeomorphism $\varphi$ is called a *coordinate chart* around $p$. A collection of charts which cover the manifold is called an *atlas*.

**Remark 1.2.** *Without the Hausdorff assumption, we would have examples such as the following: take the disjoint union* $\mathbb{R}_1 \sqcup \mathbb{R}_2$ *of two copies of the real line, and form the quotient by the equivalence relation*

$$\mathbb{R}_1 \setminus \{0\} \ni x \sim \varphi(x) \in \mathbb{R}_2 \setminus \{0\}, \tag{1}$$

*where* $\varphi$ *is the identification* $\mathbb{R}_1 \to \mathbb{R}_2$*. The resulting quotient topological space is locally homeomorphic to* $\mathbb{R}$ *but the points* $[0 \in \mathbb{R}_1], [0 \in \mathbb{R}_2]$ *cannot be separated by open neighbourhoods.*

*Second countability is not as crucial, but will be necessary for the proof of the Whitney embedding theorem, among other things.*

We now give examples of topological manifolds. The simplest is, technically, the empty set. Then we have a countable set of points (with the discrete topology), and $\mathbb{R}^n$ itself, but there are more:

**Example 1.3** (Circle)**.** Define the circle $S^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Then for any fixed point $z \in S^1$, write it as $z = e^{2\pi i c}$ for a unique real number $0 \leq c < 1$, and define the map

$$\begin{aligned} \mathbb{R} &\xrightarrow{\tilde{\nu}_z} S^1 \\ t &\longmapsto e^{2\pi i t} \end{aligned} \tag{2}$$

Let $I_c = (c - \frac{1}{2}, c + \frac{1}{2})$, and note that $\nu_z = \tilde{\nu}_z|_{I_c}$ is a homeomorphism from $I_c$ to the neighbourhood of $z$ given by $S^1 \backslash \{-z\}$. Then $\varphi_z = \nu_z^{-1}$ is a coordinate chart near $z$.

By taking products of coordinate charts, we obtain charts for the Cartesian product of manifolds. Hence the Cartesian product is a manifold.

**Example 1.4** (n-torus)**.** $S^1 \times \cdots \times S^1$ is a topological manifold (of dimension given by the number $n$ of factors), with charts $\{\varphi_{z_1} \times \cdots \times \varphi_{z_n} \ : \ z_i \in S^1\}$.

**Example 1.5** (open subsets)**.** Any open subset $U \subset M$ of a topological manifold is also a topological manifold, where the charts are simply restrictions $\varphi|_U$ of charts $\varphi$ for $M$. For instance, the real $n \times n$ matrices $\mathrm{Mat}(n, \mathbb{R})$ form a vector space isomorphic to $\mathbb{R}^{n^2}$, and contain an open subset

$$GL(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \ : \ \det A \neq 0\}, \tag{3}$$

known as the general linear group, which is a topological manifold.

**Example 1.6** (Spheres)**.** The $n$-sphere is defined as the subspace of unit vectors in $\mathbb{R}^{n+1}$:

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \ : \ \sum x_i^2 = 1\}.$$

Let $N = (1, 0, \ldots, 0)$ be the north pole and let $S = (-1, 0, \ldots, 0)$ be the south pole in $S^n$. Then we may write $S^n$ as the union $S^n = U_N \cup U_S$, where $U_N = S^n \backslash \{S\}$ and $U_S = S^n \backslash \{N\}$ are equipped with coordinate charts $\varphi_N, \varphi_S$ into $\mathbb{R}^n$, given by the "stereographic projections" from the points $S, N$ respectively

$$\varphi_N : (x_0, \vec{x}) \mapsto (1 + x_0)^{-1} \vec{x}, \tag{4}$$

$$\varphi_S : (x_0, \vec{x}) \mapsto (1 - x_0)^{-1} \vec{x}. \tag{5}$$

**Remark 1.7.** *We have endowed the sphere $S^n$ with a certain topology, but is it possible for another topological manifold $\tilde{S}^n$ to be homotopy equivalent to $S^n$* without *being homeomorphic to it? The answer is no, and this is known as the topological Poincaré conjecture, and is usually stated as follows: any homotopy n-sphere is homeomorphic to the n-sphere. It was proven for $n > 4$ by Smale, for $n = 4$ by Freedman, and for $n = 3$ is equivalent to the smooth Poincaré conjecture which was proved by Hamilton-Perelman. In dimensions $n = 1, 2$ it is a consequence of the classification of topological 1- and 2-manifolds.*

**Example 1.8** (Projective spaces)**.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $\mathbb{K}P^n$ is defined to be the space of lines through $\{0\}$ in $\mathbb{K}^{n+1}$, and is called the projective space over $\mathbb{K}$ of dimension $n$.

More precisely, let $X = \mathbb{K}^{n+1} \backslash \{0\}$ and define an equivalence relation on $X$ via $x \sim y$ iff $\exists \lambda \in \mathbb{K}^* = \mathbb{K} \backslash \{0\}$ such that $\lambda x = y$, i.e. $x, y$ lie on the same line through the origin. Then

$$\mathbb{K}P^n = X / \sim,$$

and it is equipped with the quotient topology.

The projection map $\pi : X \longrightarrow \mathbb{K}P^n$ is an *open* map, since if $U \subset X$ is open, then $tU$ is also open $\forall t \in \mathbb{K}^*$, implying that $\cup_{t \in \mathbb{K}^*} tU = \pi^{-1}(\pi(U))$ is open, implying $\pi(U)$ is open. This immediately shows, by the way, that $\mathbb{K}P^n$ is second countable.

To show $\mathbb{K}P^n$ is Hausdorff (which we must do, since Hausdorff is preserved by subspaces and products, but *not* quotients), we show that the graph of the equivalence relation is closed in $X \times X$ (this, together with the openness of $\pi$, gives us the Hausdorff property for $\mathbb{K}P^n$). This graph is simply

$$\Gamma_\sim = \{(x, y) \in X \times X \ : \ x \sim y\},$$

and we notice that $\Gamma_\sim$ is actually the common zero set of the following continuous functions

$$f_{ij}(x,y) = (x_i y_j - x_j y_i) \quad i \neq j.$$

An atlas for $\mathbb{K}P^n$ is given by the open sets $U_i = \pi(\tilde{U}_i)$, where

$$\tilde{U}_i = \{(x_0, \ldots, x_n) \in X \ : \ x_i \neq 0\},$$

and these are equipped with charts to $\mathbb{K}^n$ given by

$$\varphi_i([x_0, \ldots, x_n]) = x_i^{-1}(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \tag{6}$$

which are indeed invertible by $(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n)$.

Sometimes one finds it useful to simply use the "coordinates" $(x_0, \ldots, x_n)$ for $\mathbb{K}P^n$, with the understanding that the $x_i$ are well-defined only up to overall rescaling. This is called using "projective coordinates" and in this case a point in $\mathbb{K}P^n$ is denoted by $[x_0 : \cdots : x_n]$.

**Example 1.9** (Connected sum)**.** Let $p \in M$ and $q \in N$ be points in topological manifolds and let $(U, \varphi)$ and $(V, \psi)$ be charts around $p, q$ such that $\varphi(p) = 0$ and $\psi(q) = 0$.

Choose $\epsilon$ small enough so that $B(0, 2\epsilon) \subset \varphi(U)$ and $B(0, 2\epsilon) \subset \varphi(V)$, and define the map of annuli

$$\begin{array}{ccc} B(0,2\epsilon)\backslash\overline{B(0,\epsilon)} & \xrightarrow{\phi} & B(0,2\epsilon)\backslash\overline{B(0,\epsilon)} \\ x & \longmapsto & \frac{2\epsilon^2}{|x|^2}x \end{array} \tag{7}$$

This is a homeomorphism of the annulus to itself, exchanging the boundaries. Now we define a new topological manifold, called the *connected sum* $M\#N$, as the quotient $X/\sim$, where

$$X = (M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}) \sqcup (N\backslash\overline{\psi^{-1}(B(0,\epsilon))}),$$

and we define an identification $x \sim \psi^{-1}\phi\varphi(x)$ for $x \in \varphi^{-1}(B(0, 2\epsilon))$. If $\mathcal{A}_M$ and $\mathcal{A}_N$ are atlases for $M, N$ respectively, then a new atlas for the connect sum is simply

$$\mathcal{A}_M|_{M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}} \cup \mathcal{A}_N|_{N\backslash\overline{\psi^{-1}(B(0,\epsilon))}}$$

**Remark 1.10.** *The homeomorphism type of the connected sum of connected manifolds $M, N$ is independent of the choices of $p, q$ and $\varphi, \psi$, except that it may depend on the two possible orientations of the gluing map $\psi^{-1}\phi\varphi$. To prove this, one must appeal to the so-called* annulus theorem*.*

**Remark 1.11.** *By iterated connect sum of $S^2$ with $T^2$ and $\mathbb{R}P^2$, we can obtain all compact 2-dimensional manifolds.*

**Example 1.12.** Let $F$ be a topological space. A fiber bundle with fiber $F$ is a triple $(E, p, B)$, where $E, B$ are topological spaces called the "total space" and "base", respectively, and $p : E \longrightarrow B$ is a continuous surjective map called the "projection map", such that, for each point $b \in B$, there is a neighbourhood $U$ of $b$ and a homeomorphism

$$\Phi : p^{-1}U \longrightarrow U \times F,$$

such that $p_U \circ \Phi = p$, where $p_U : U \times F \longrightarrow U$ is the usual projection. The submanifold $p^{-1}(b) \cong F$ is called the "fiber over $b$".

When $B, F$ are topological manifolds, then clearly $E$ becomes one as well. We will often encounter such manifolds.

**Example 1.13** (General gluing construction)**.** To construct a topological manifold "from scratch", we glue open subsets of $\mathbb{R}^n$ together using homeomorphisms, as follows.

Begin with a countable collection of open subsets of $\mathbb{R}^n$: $\mathcal{A} = \{U_i\}$. Then for each $i$, we choose finitely many open subsets $U_{ij} \subset U_i$ and gluing maps

$$U_{ij} \xrightarrow{\varphi_{ij}} U_{ji}\ , \tag{8}$$

which we require to satisfy $\varphi_{ij}\varphi_{ji} = \mathrm{Id}_{U_{ji}}$, and such that $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all $k$, and most important of all, $\varphi_{ij}$ must be *homeomorphisms*.

Next, we want the pairwise gluings to be consistent (transitive) and so we require that $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \mathrm{Id}_{U_{ij} \cap U_{jk}}$ for all $i, j, k$. This will ensure that the equivalence relation in (10) is well-defined.

Second countability of the glued manifold is guaranteed since we started with a countable collection of opens, but the Hausdorff property is not necessarily satisfied without a further assumption: we require that the graph of $\varphi_{ij}$, namely

$$\{(x, \varphi_{ij}(x))\ : x \in U_{ij}\} \tag{9}$$

is a closed subset of $U_i \times U_j$.

The final glued topological manifold is then

$$M = \frac{\bigsqcup U_i}{\sim}, \tag{10}$$

for the equivalence relation $x \sim \varphi_{ij}(x)$ for $x \in U_{ij}$, for all $i, j$. This space has a distinguished atlas $\mathcal{A}$, whose charts are simply the inclusions of the $U_i$ in $\mathbb{R}^n$.