1.5 Smooth maps

For topological manifolds M, N of dimension m, n, the natural notion of morphism from M to N is that of a continuous map. A continuous map with continuous inverse is then a homeomorphism from M to N, which is the natural notion of equivalence for topological manifolds. Since the composition of continuous maps is continuous, we obtain a "category" of topological manifolds and continuous maps.

A category is a class of objects $\mathcal{C}$ (in our case, topological manifolds) and a class of arrows $\mathcal{A}$ (in our case, continuous maps). Each arrow goes from an object (the source) to another object (the target), meaning that there are "source" and "target" maps from $\mathcal{A}$ to $\mathcal{C}$:

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \mathcal{C} \end{tikzcd} \tag{25}$$

Also, a category has an identity arrow for each object, given by a map $\mathrm{id} : \mathcal{C} \longrightarrow \mathcal{A}$ (in our case, the identity map of any manifold to itself). Furthermore, there is an associative composition operation on arrows.

Conventionally, we write the set of arrows from X to X as $\mathrm{Hom}(X, Y)$, i.e.

$$\mathrm{Hom}(X, Y) = \{a \in \mathcal{A} \ : \ s(a) = X \text{ and } t(a) = Y\}. \tag{26}$$

Then the associative composition of arrows mentioned above becomes a map

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z). \tag{27}$$

We have described the category of topological manifolds; we now describe the category of smooth manifolds by defining the notion of a smooth map.

Definition 1.32. A map $f : M \to N$ is called *smooth* when for each chart (U, φ) for M and each chart (V, ψ) for N, the composition $\psi \circ f \circ \varphi^{-1}$ is a smooth map, i.e. $\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$.

The set of smooth maps (i.e. morphisms) from M to N is denoted $C^\infty(M, N)$. A smooth map with a smooth inverse is called a *diffeomorphism*.

Proposition 1.33. *If $g : L \to M$ and $f : M \to N$ are smooth maps, then so is the composition $f \circ g$.*

Proof. If charts φ, χ, ψ for L, M, N are chosen near $p \in L$, $g(p) \in M$, and $(fg)(p) \in N$, then $\psi \circ (f \circ g) \circ \varphi^{-1} = A \circ B$, for $A = \psi f \chi^{-1}$ and $B = \chi g \varphi^{-1}$ both smooth mappings $\mathbb{R}^n \to \mathbb{R}^n$. By the chain rule, $A \circ B$ is differentiable at p, with derivative $D_p(A \circ B) = (D_{g(p)}A)(D_pB)$ (matrix multiplication). $\square$

Now we have a new category, the category of smooth manifolds and smooth maps; two manifolds are considered isomorphic when they are diffeomorphic. In fact, the definitions above carry over, word for word, to the setting of manifolds with boundary. Hence we have defined another category, the category of smooth manifolds with boundary.

In defining the arrows for the category of manifolds with boundary, we may choose to consider all smooth maps, or only those smooth maps which send the boundary to the boundary, i.e. boundary-preserving maps.

The operation ∂ of "taking the boundary" sends a manifold with boundary to a usual manifold. Furthermore, if $\psi : M \to N$ is a boundary-preserving smooth map, then we can "take its boundary" by restricting it to the boundary, i.e. $\partial\psi = \psi|_{\partial M}$. Since ∂ takes objects to objects and arrows to arrows in a manner which respects compositions and identity maps, it is called a "functor" from the category of manifolds with boundary (and boundary-preserving smooth maps) to the category of smooth manifolds.

Example 1.34. Let φ_z be a chart for S^1, and let $j : S^1 \to \mathbb{C}$ be the inclusion map of S^1. We see that j is smooth since $j \circ \varphi^{-1}$ is the map

$$t \mapsto e^{2\pi i t} = (\cos(2\pi t), \sin(2\pi t)), \tag{28}$$

which is a smooth map from $I_c \subset \mathbb{R}$ to $\mathbb{R}^2$.

Example 1.35. The complex projective line $\mathbb{C}P^1$ is diffeomorphic to the 2-sphere S^2: consider the maps $f_+(x_0, x_1, x_2) = [1+x_0 : x_1+ix_2]$ and $f_-(x_0, x_1, x_2) = [x_1 - ix_2 : 1 - x_0]$. Since $f_\pm$ is continuous on $x_0 \neq \pm 1$, and since $f_- = f_+$ on $|x_0| < 1$, the pair (f_-, f_+) defines a continuous map $f : S^2 \longrightarrow \mathbb{C}P^1$. To check smoothness, we compute the compositions

$$\varphi_0 \circ f_+ \circ \varphi_N^{-1} : (y_1, y_2) \mapsto y_1 + iy_2, \tag{29}$$

$$\varphi_1 \circ f_- \circ \varphi_S^{-1} : (y_1, y_2) \mapsto y_1 - iy_2, \tag{30}$$

both of which are obviously smooth maps.

Example 1.36. The smooth inclusion $j : S^1 \to \mathbb{C}$ induces a smooth inclusion $j \times j$ of the 2-torus $T^2 = S^1 \times S^1$ into $\mathbb{C}^2$. The image of $j \times j$ does not include zero, so we may compose with the projection $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$ and the diffeomorphism $\mathbb{C}P^1 \to S^2$, to obtain a smooth map

$$\pi \circ (j \times j) : T^2 \to S^2. \tag{31}$$

Remark 1.37 (Exotic smooth structures)**.** The topological Poincaré conjecture, now proven, states that any topological manifold homotopic to the n-sphere is in fact homeomorphic to it. We have now seen how to put a differentiable structure on this n-sphere. Remarkably, there are other differentiable structures on the n-sphere which are not diffeomorphic to the standard one we gave; these are called *exotic* spheres.

Since the connected sum of spheres is homeomorphic to a sphere, and since the connected sum operation is well-defined as a smooth manifold, it follows that the connected sum defines a *monoid* structure on the set of smooth n-spheres. In fact, Kervaire and Milnor showed that for $n \neq 4$, the set of (oriented) diffeomorphism classes of smooth n-spheres forms a finite abelian group under the connected sum operation. This is not known to be the case in four dimensions.

Kervaire and Milnor also compute the order of this group, and the first dimension where there is more than one smooth sphere is $n = 7$, in which case they show there are 28 smooth spheres, which we will encounter later on.

The situation for spheres may be contrasted with that for the Euclidean spaces: any differentiable manifold homeomorphic to $\mathbb{R}^n$ for $n \neq 4$ must be diffeomorphic to it. On the other hand, by results of Donaldson, Freedman, Taubes, and Kirby, we know that there are uncountably many non-diffeomorphic smooth structures on the topological manifold $\mathbb{R}^4$; these are called *fake* $\mathbb{R}^4$s.

Remark 1.38. The maps $\alpha : x \mapsto x$ and $\beta : x \mapsto x^3$ are both homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$. Each one defines, by itself, a smooth atlas on $\mathbb{R}$. These two smooth atlases are not compatible (why?), so they do not define the same smooth structure on $\mathbb{R}$. Nevertheless, the smooth structures are equivalent, since there is a diffeomorphism taking one to the other. What is it?

Example 1.39 (Lie groups)**.** A group is a set G with an associative multiplication $G \times G \xrightarrow{m} G$, an identity element $e \in G$, and an inversion map $\iota : G \longrightarrow G$, usually written $\iota(g) = g^{-1}$.

If we endow G with a topology for which G is a topological manifold and m, ι are continuous maps, then the resulting structure is called a *topological group*. If G is a given a smooth structure and m, ι are smooth maps, the result is a *Lie group*.

The real line (where m is given by addition), the circle (where m is given by complex multiplication), and their Cartesian products give simple but important examples of Lie groups. We have also seen the general linear group $GL(n, \mathbb{R})$, which is a Lie group since matrix multiplication and inversion are smooth maps.

Since $m : G \times G \longrightarrow G$ is a smooth map, we may fix $g \in G$ and define smooth maps $L_g : G \longrightarrow G$ and $R_g : G \longrightarrow G$ via $L_g(h) = gh$ and $R_g(h) = hg$. These are called *left multiplication* and *right multiplication*. Note that the group axioms imply that $R_g L_h = L_h R_g$.

2 The tangent bundle

The tangent bundle of a manifold is an absolutely central topic in differential geometry. In this section, we describe the tangent bundle intrinsically, without reference to any embedding of the manifold in a vector space. By way of motivation, however, we briefly discuss this case.

The definition of the tangent bundle is simplest for an open subset $U \subset V$ of a finite-dimensional vector space V. In this case, a tangent vector to a point $p \in U$ is simply a vector in V, and so the tangent bundle, which consists of all tangent vectors to all points in U, is simply given by

$$TU = U \times V. \tag{32}$$

The tangent bundle TU of U is then equipped with a projection map $\pi : TU \to U$, and a vector field on U is nothing but a *section* of this projection, i.e.

a smooth map $X : U \to TU$ such that $\pi \circ X = \mathrm{id}_U$. We now investigate the problem of generalizing the tangent bundle to other manifolds, where the convenience of being an open set in a vector space is not available.

2.1 Submanifolds of Euclidean space

There are several ways to define the notion of submanifold. We will use a very basic definition which works for topological and smooth manifolds and which is based on the local model of inclusion of a vector subspace. These are sometimes called *regular* or *embedded* submanifolds.

Definition 2.1. A subspace $L \subset M$ of an m-manifold is called a submanifold of codimension k when each point $x \in L$ is contained in a chart (U, φ) for M such that

$$L \cap U = f^{-1}(0), \tag{33}$$

where f is the composition of φ with the projection $\mathbb{R}^m \to \mathbb{R}^k$ to the last k coordinates $(x_{m-k+1}, \ldots, x_m)$. A submanifold of codimension 1 is usually called a *hypersurface*.

Now suppose that $L \subset \mathbb{R}^m$ is a submanifold of codimension k, and let φ be a diffeomorphism which "rectifies" a neighbourhood $U \subset \mathbb{R}^n$ of a point $p \in L$, sending U to an open set in $\mathbb{R}^m$ in which the image of $L \cap U$ is a linear subspace, given by $x_{m-k+1} = \cdots = x_m = 0$. Then we say that $u \in \mathbb{R}^m$ is tangent to L at p when the derivative $D\varphi(p)$ takes u to that same linear subspace.

The tangent bundle TL of L is the set of all pairs (p, u), where $p \in L$ and $u \in \mathbb{R}^m$ is tangent to L at p. It is a subset of $T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$, and one can prove that it is a submanifold of $\mathbb{R}^{2m}$ of codimension $2k$.

2.2 General construction

The tangent bundle of an n-manifold M is a $2n$-manifold, called TM, naturally constructed in terms of M. As a set, it is fairly easy to describe, as simply the disjoint union of all tangent spaces. However we must explain precisely what we mean by the tangent space T_pM to $p \in M$.

Definition 2.2. Let $(U, \varphi), (V, \psi)$ be coordinate charts around $p \in M$. Let $u \in T_{\varphi(p)}\varphi(U)$ and $v \in T_{\psi(p)}\psi(V)$. Then the triples (U, φ, u), (V, ψ, v) are called equivalent when $D(\psi \circ \varphi^{-1})(\varphi(p)) : u \mapsto v$. The chain rule for derivatives $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ guarantees that this is indeed an equivalence relation.

The set of equivalence classes of such triples is called the tangent space to p of M, denoted T_pM, and forms a real vector space of dimension $\dim M$.

As a set, the tangent bundle is defined by

$$TM = \bigsqcup_{p \in M} T_pM,$$

and it is equipped with a natural surjective map $\pi : TM \longrightarrow M$, which is simply $\pi(X) = x$ for $X \in T_xM$.

We now give it a manifold structure in a natural way.

Proposition 2.3. *For an n-manifold M, the set TM has a natural topology and smooth structure which make it a $2n$-manifold, and make $\pi : TM \longrightarrow M$ a smooth map.*

Proof. Any chart (U, φ) for M defines a bijection

$$T\varphi(U) \cong U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U) \tag{34}$$

via $(p, v) \mapsto (U, \varphi, v)$. Using this, we induce a smooth manifold structure on $\pi^{-1}(U)$, and view the inverse of this map as a chart $(\pi^{-1}(U), \Phi)$ to $\varphi(U) \times \mathbb{R}^n$.

given another chart (V, ψ), we obtain another chart $(\pi^{-1}(V), \Psi)$ and we may compare them via

$$\Psi \circ \Phi^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n, \tag{35}$$

which is given by $(p, u) \mapsto ((\psi \circ \varphi^{-1})(p), D(\psi \circ \varphi^{-1})_p u)$, which is smooth. Therefore we obtain a topology and smooth structure on all of TM (by defining W to be open when $W \cap \pi^{-1}(U)$ is open for every U in an atlas for M; all that remains is to verify the Hausdorff property, which holds since points x, y are either in the same chart (in which case it is obvious) or they can be separated by the given type of charts. $\square$

Remark 2.4. This is a more constructive way of looking at the tangent bundle: We choose a countable, locally finite atlas $\{(U_i, \varphi_i)\}$ for M and glue together $U_i \times \mathbb{R}^n$ to $U_j \times \mathbb{R}^n$ via an equivalence

$$(x, u) \sim (y, v) \;\Leftrightarrow\; y = \varphi_j \circ \varphi_i^{-1}(x) \;\text{ and }\; v = D(\varphi_j \circ \varphi_i^{-1})_x u, \tag{36}$$

and verify the conditions of the general gluing construction 1.13. Then show that a different atlas gives a canonically diffeomorphic manifold, i.e. that the result is independent of atlas.