Exercise 1. Construct, using the stereographic charts for S^2 given in class, a smooth vector field on S^2 which vanishes exactly at 2 points, and another vector field which vanishes at exactly 1 point.

Exercise 2. Consider the Hopf map $p : S^3 \to S^2$ given by composing the embedding $S^3 \subset \mathbb{C}^2\backslash\{0\}$ with the projection $\pi : \mathbb{C}^2\backslash\{0\} \to \mathbb{C}P^1 \cong S^2$. In the first assignment, you computed this map explicitly in the given coordinate charts for S^3 and S^2. Now compute the derivative of this map in two distinct sets of charts and verify that these derivatives are indeed glued together by the gluing data for the tangent bundles of S^3 and S^2.

Exercise 3.

1. Prove that the orthogonal group $O(n) = \{X \in GL(n, \mathbb{R}) : XX^\top = 1\}$ is a smooth submanifold of $M(n, \mathbb{R})$, the $n \times n$ matrices. To show this, consider the map $f : M(n, \mathbb{R}) \to S(n, \mathbb{R})$ to the symmetric matrices $S(n, \mathbb{R})$ given by $f(X) = XX^\top$.
2. Prove that the special orthogonal group $SO(n, \mathbb{R})$ is a smooth submanifold of $M(n, \mathbb{R})$.
3. Prove that the unitary and special unitary groups $U(n)$, $SU(n)$ are smooth submanifolds of $M(n, \mathbb{C})$.
4. Prove that $SU(2)$ is diffeomorphic to S^3 and that $U(2)$ is diffeomorphic to $S^3 \times S^1$.

Exercise 4.

1. Consider the subset of $T\mathbb{R}^3$ consisting of the vectors tangent to the 2-sphere $S^2 \subset \mathbb{R}^3$ and of unit length (we use the usual Euclidean length on $\mathbb{R}^3$, and the fact that $T\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$). Prove this subset is a submanifold, and prove it is diffeomorphic to $SO(3)$.
2. Deduce from the above that $SO(3)$ is null-cobordant.

In fact $SO(3)$ is isomorphic to $SU(2)/\{\pm 1\}$ and so is diffeomorphic to $S^3/\mathbb{Z}_2 \cong \mathbb{R}P^3$, so the above shows that $\mathbb{R}P^3$ is null-cobordant, explaining the exclusion of x_3 in Thom's theorem.

Exercise 5. Prove that if K is a submanifold[1] of L and L is a submanifold of M, then K is a submanifold of M.

Exercise 6. Let K, K' be transverse submanifolds of codimension k, k' in the n-manifold M. Prove that each point $p \in K \cap K'$ has a neighbourhood $U \subset M$ and a diffeomorphism from U to a neighbourhood of the origin in $\mathbb{R}^n$ which takes K and K' to the coordinate planes $V(x_1, \ldots, x_k)$ and $V(x_{n-k'+1}, \ldots, x_n)$, respectively (Here V denotes the common zero set of its arguments).

Exercise 7. Let $f : M \to M$ be a smooth map and suppose p is a fixed point of f, i.e. $f(p) = p$. The point p is called a *Lefschetz fixed point* when the derivative map $f_* : T_pM \to T_pM$ does not have $+1$ as an eigenvalue.

Show that if M is compact and all fixed points for f are Lefschetz, then there are only finitely many fixed points for f.

[1] We always mean *embedded* or *regular* submanifold when we say submanifold.