Exercise 1. Fix a positive integer k and complex numbers $c_0, c_1, \ldots, c_{k-1}$. Consider the map $f : \mathbb{C} \to \mathbb{C}$ given by

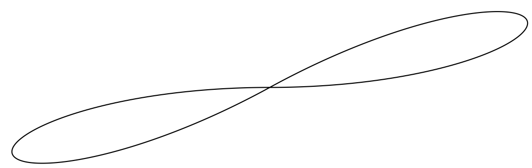
$$f(z) = z^k + c_{k-1}z^{k-1} + \cdots + c_1 z + c_0.$$

1. Identifying $\mathbb{C}$ with the standard affine chart U_0 for $\mathbb{C}P^1$, prove that f extends uniquely to a smooth map from $\mathbb{C}P^1$ to $\mathbb{C}P^1$. Call this map $\tilde{f}$.

2. Determine the degree modulo 2 of $\tilde{f}$.

3. If k is odd, can you show that $\tilde{f}$ is surjective? What does this imply about the number of zeros of f in this case? How is this related to the fundamental theorem of algebra?

Exercise 2. Consider an immersion $i : S^1 \to \mathbb{R}^2$ following a "figure eight path" as shown below.



Prove that there is no smooth homotopy from i to the standard embedding $j : S^1 \to \mathbb{R}^2$ which remains an immersion at all intermediate times.

Exercise 3. Let $f : M \to \mathbb{R}$ be a proper submersion. Then $V = \ker Df$ defines a codimension 1 subbundle of TM called the vertical bundle.

1. Show, using a partition of unity, that it is possible to choose a rank 1 subbundle $H \subset TM$ complementary to V. Do not use a Riemannian metric.

2. Conclude that to any vector field v on $\mathbb{R}$ we may associate a unique vector field v^h on M which lies in H. This is called the horizontal lift of v.

3. Prove the preimages of any pair of points in the image of f are diffeomorphic manifolds.

Exercise 4. Let V be a finite dimensional vector space, and view it as a manifold M. Then $TM = V \times V$.

1. The trivial map $E : x \mapsto (x, x)$ defines a section of the tangent bundle, i.e. a vector field. Compute the time-t flow of this vector field and determine whether it is complete or not.

2. Suppose $A : V \to V$ is a linear map. Then the map $A : x \mapsto (x, Ax)$ defines a vector field on M; compute its flow and determine if it is complete.

3. If A, B are linear maps as above, compute the Lie derivative of the vector fields they determine. Verify the fact that if the vector fields commute, then the flows commute.