Exercise 1. Determine the flow of the vector field $E = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ on $\mathbb{R}^n$. Deduce that this vector field is independent of the choice of linear coordinates.

Exercise 2. Consider S^n and its two stereographic coordinate charts φ_S, φ_N to $\mathbb{R}^n$. Using standard coordinates on $\mathbb{R}^n$, write down the coordinate expressions for a smooth, nowhere-vanishing n-form on S^n.

Exercise 3. Let X be a vector field. Define the operator (the *interior product*) $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ by the formula

$$(i_X\rho)(X_1, \ldots X_{k-1}) = \rho(X, X_1, \ldots, X_{k-1}).$$

1. Prove that i_X is a graded derivation of degree -1 of the algebra of differential forms.
2. Prove that the graded commutator $[D_1, D_2] = D_1D_2 - (-1)^{d_1d_2}D_2D_1$ of the graded derivations D_1, D_2 of degree d_1, d_2 is itself a graded derivation of degree $d_1 + d_2$.
3. Define $L_X = di_X + i_Xd$ (the *Lie derivative*) and prove it is a graded derivation of degree zero.
4. Prove the identity $i_{[X,Y]} = [L_X, i_Y]$, where the left bracket is the Lie bracket of vector fields and the right bracket is the graded commutator. Note that this identity expresses the intimate relationship between the Lie bracket of vector fields (on the left hand side) and the exterior derivative (on the right hand side).
5. Prove the identity $L_{[X,Y]} = [L_X, L_Y]$.

Exercise 4. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$(r, \phi, \theta) \mapsto (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi),$$

where (r, ϕ, θ) are standard Cartesian coordinates on $\mathbb{R}^3$.

- Compute $\varphi^*dx, \varphi^*dy, \varphi^*dz$ where (x, y, z) are Cartesian coordinates for $\mathbb{R}^3$.
- Compute $\varphi^*(dx \wedge dy \wedge dz)$.
- Compute the integral
$$\int_{S_r^2} i_X(dx \wedge dy \wedge dz),$$
for the vector field $X = \varphi_* \frac{\partial}{\partial r}$, where S_r^2 is the sphere of radius r.

Exercise 5. Use Stokes' theorem if necessary:

1. Let M be a compact orientable smooth n-manifold (without boundary) and let $\mu \in \Omega^{n-1}(M)$. Prove there exists a point $p \in M$ with $d\mu(p) = 0$.

2. For any sphere S^k, let $\iota : S^k \to \mathbb{R}^{k+1}$ be the usual inclusion, and let $v_k \in \Omega^k(S^k)$ be given by

$$v_k = \iota^* \sum_{i=0}^{k} (-1)^i x^i dx^0 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^k.$$

   Show that v_k is closed and that $[v_k] \neq 0$ in the top de Rham cohomology group $H^k(S^k)$.

Exercise 6. Compute the de Rham cohomology groups (Using Mayer-Vietoris if necessary) of the following spaces, for all degrees.

- $\mathbb{R}^3 - \{p\}$, for $p \in \mathbb{R}^3$ a point.
- $\mathbb{R}^3 - \{p_1 \cup p_2\}$ where p_i are distinct points?
- $\mathbb{R}^3 - \{\ell_1 \cup \ell_2\}$ where ℓ_i are non-intersecting lines?
- $\mathbb{R}^3 - \{\ell_1 \cup \ell_2\}$, assuming that l_1 intersects l_2 in exactly one point?

This question is a slightly easier version of the one John Nash asked in class in the movie "A Beautiful Mind".