1 Manifolds

A manifold is a space which looks like $\mathbb{R}^n$ at small scales (i.e. "locally"), but which may be very different from this at large scales (i.e. "globally"). In other words, manifolds are made by gluing pieces of $\mathbb{R}^n$ together to make a more complicated whole. We want to make this precise.

1.1 Topological manifolds

Definition 1.1. A real, n-dimensional *topological manifold* is a Hausdorff, second countable topological space which is locally homeomorphic to $\mathbb{R}^n$.

"Locally homeomorphic to $\mathbb{R}^n$" simply means that each point p has an open neighbourhood U for which we can find a homeomorphism $\varphi : U \longrightarrow V$ to an open subset $V \in \mathbb{R}^n$. Such a homeomorphism φ is called a *coordinate chart* around p. A collection of charts which cover the manifold is called an *atlas*.

We now give examples of topological manifolds. The simplest is, technically, the empty set. Then we have a countable set of points (with the discrete topology), and $\mathbb{R}^n$ itself, but there are more:

Example 1.2 (open subsets)**.** Any open subset $U \subset M$ of a topological manifold is also a topological manifold, where the charts are simply restrictions $\varphi|_U$ of charts φ for M. For instance, the real $n \times n$ matrices $\mathrm{Mat}(n, \mathbb{R})$ form a vector space isomorphic to $\mathbb{R}^{n^2}$, and contain an open subset

$$GL(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \ : \ \det A \neq 0\}, \tag{1}$$

known as the general linear group, which is a topological manifold.

Example 1.3 (Circle)**.** The circle is defined as the subspace of unit vectors in $\mathbb{R}^2$:

$$S^1 = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}.$$

Let $N = (0, 1)$ be the north pole and let $S = (0, -1)$ be the south pole in S^n. Then we may write S^n as the union $S^n = U_N \cup U_S$, where $U_N = S^n \backslash \{N\}$ and $U_S = S^n \backslash \{S\}$ are equipped with coordinate charts φ_N, φ_S into $\mathbb{R}^n$, given by the "stereographic projections" from the points S, N respectively

$$\varphi_N : (x, y) \mapsto (1 - y)^{-1} x, \tag{2}$$

$$\varphi_S : (x, y) \mapsto (1 + y)^{-1} x. \tag{3}$$

By taking products of coordinate charts, we obtain charts for the Cartesian product of manifolds. Hence the Cartesian product is a manifold.

Example 1.4 (n-torus)**.** $S^1 \times \cdots \times S^1$ is a topological manifold (of dimension given by the number n of factors), with charts $\{\varphi_{z_1} \times \cdots \times \varphi_{z_n} \ : \ z_i \in S^1\}$.

The circle is a 1-dimensional sphere; we now describe general spheres.

Example 1.5 (Spheres)**.** The n-sphere is defined as the subspace of unit vectors in $\mathbb{R}^{n+1}$:

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \ : \ \sum x_i^2 = 1\}.$$

Let $N = (1, 0, \dots, 0)$ be the north pole and let $S = (-1, 0, \dots, 0)$ be the south pole in S^n. Then we may write S^n as the union $S^n = U_N \cup U_S$, where $U_N = S^n \backslash \{N\}$ and $U_S = S^n \backslash \{S\}$ are equipped with coordinate charts φ_N, φ_S into $\mathbb{R}^n$, given by the "stereographic projections" from the points S, N respectively

$$\varphi_N : (x_0, \vec{x}) \mapsto (1 - x_0)^{-1}\vec{x}, \tag{4}$$

$$\varphi_S : (x_0, \vec{x}) \mapsto (1 + x_0)^{-1}\vec{x}. \tag{5}$$

Remark 1.6. We have endowed the sphere S^n with a certain topology, but is it possible for another topological manifold $\tilde{S}^n$ to be homotopy equivalent to S^n *without* being homeomorphic to it? The answer is no, and this is known as the topological Poincaré conjecture, and is usually stated as follows: any homotopy n-sphere is homeomorphic to the n-sphere. It was proven for $n > 4$ by Smale, for $n = 4$ by Freedman, and for $n = 3$ is equivalent to the smooth Poincaré conjecture which was proved by Hamilton-Perelman. In dimensions $n = 1, 2$ it is a consequence of the classification of topological 1- and 2-manifolds.

Remark 1.7 (The Hausdorff and second countability axioms)**.** Without the Hausdorff assumption, we would have examples such as the following: take the disjoint union $\mathbb{R}_1 \sqcup \mathbb{R}_2$ of two copies of the real line, and form the quotient by the equivalence relation

$$\mathbb{R}_1 \setminus \{0\} \ni x \sim \varphi(x) \in \mathbb{R}_2 \setminus \{0\}, \tag{6}$$

where φ is the identification $\mathbb{R}_1 \to \mathbb{R}_2$. The resulting quotient topological space is locally homeomorphic to $\mathbb{R}$ but the points $[0 \in \mathbb{R}_1], [0 \in \mathbb{R}_2]$ cannot be separated by open neighbourhoods.

Second countability is not as crucial, but will be necessary for the proof of the Whitney embedding theorem, among other things.

Example 1.8 (Projective spaces)**.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $\mathbb{K}P^n$ is defined to be the space of lines through $\{0\}$ in $\mathbb{K}^{n+1}$, and is called the projective space over $\mathbb{K}$ of dimension n.

More precisely, let $X = \mathbb{K}^{n+1} \backslash \{0\}$ and define an equivalence relation on X via $x \sim y$ iff $\exists \lambda \in \mathbb{K}^* = \mathbb{K} \backslash \{0\}$ such that $\lambda x = y$, i.e. x, y lie on the same line through the origin. Then

$$\mathbb{K}P^n = X / \sim,$$

and it is equipped with the quotient topology.

The projection map $\pi : X \longrightarrow \mathbb{K}P^n$ is an *open* map, since if $U \subset X$ is open, then tU is also open $\forall t \in \mathbb{K}^*$, implying that $\cup_{t \in \mathbb{K}^*} tU = \pi^{-1}(\pi(U))$ is open, implying $\pi(U)$ is open. This immediately shows, by the way, that $\mathbb{K}P^n$ is second countable.

To show $\mathbb{K}P^n$ is Hausdorff (which we must do, since Hausdorff is preserved by subspaces and products, but *not* quotients), we show that the graph of the equivalence relation is closed in $X \times X$ (this, together with the openness of π, gives us the Hausdorff property for $\mathbb{K}P^n$). This graph is simply

$$\Gamma_\sim = \{(x,y) \in X \times X \ : \ x \sim y\},$$

and we notice that $\Gamma_\sim$ is actually the common zero set of the following continuous functions

$$f_{ij}(x,y) = (x_i y_j - x_j y_i) \quad i \neq j,$$

implying at once that it is a closed subset.

An atlas for $\mathbb{K}P^n$ is given by the open sets $U_i = \pi(\tilde{U}_i)$, where

$$\tilde{U}_i = \{(x_0, \ldots, x_n) \in X \ : \ x_i \neq 0\},$$

and these are equipped with charts to $\mathbb{K}^n$ given by

$$\varphi_i([x_0, \ldots, x_n]) = x_i^{-1}(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \tag{7}$$

which are indeed invertible by $(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n)$.

Sometimes one finds it useful to simply use the "coordinates" $(x_0, \ldots, x_n)$ for $\mathbb{K}P^n$, with the understanding that the x_i are well-defined only up to overall rescaling. This is called using "projective coordinates" and in this case a point in $\mathbb{K}P^n$ is denoted by $[x_0 : \cdots : x_n]$.

Example 1.9 (Connected sum)**.** Let $p \in M$ and $q \in N$ be points in topological manifolds and let (U, φ) and (V, ψ) be charts around p, q such that $\varphi(p) = 0$ and $\psi(q) = 0$.

Choose ϵ small enough so that $B(0, 2\epsilon) \subset \varphi(U)$ and $B(0, 2\epsilon) \subset \varphi(V)$, and define the map of annuli

$$\begin{aligned} B(0,2\epsilon)\backslash\overline{B(0,\epsilon)} &\xrightarrow{\ \phi\ } B(0,2\epsilon)\backslash\overline{B(0,\epsilon)} \\ x &\longmapsto \tfrac{2\epsilon^2}{|x|^2} x \end{aligned} \tag{8}$$

This is a homeomorphism of the annulus to itself, exchanging the boundaries. Now we define a new topological manifold, called the *connected sum* $M\#N$, as the quotient $X/\sim$, where

$$X = (M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}) \sqcup (N\backslash\overline{\psi^{-1}(B(0,\epsilon))}),$$

and we define an identification $x \sim \psi^{-1}\phi\varphi(x)$ for $x \in \varphi^{-1}(B(0, 2\epsilon))$. If $\mathcal{A}_M$ and $\mathcal{A}_N$ are atlases for M, N respectively, then a new atlas for the connect sum is simply

$$\mathcal{A}_M|_{M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}} \cup \mathcal{A}_N|_{N\backslash\overline{\psi^{-1}(B(0,\epsilon))}}.$$

Remark 1.10. The connected sum operation as described above may be viewed as an operation on the pair $(L, \{p, q\})$, where $L = M \sqcup N$ is the manifold formed by the disjoint union of M and N and $\{p, q\} \subset L$ is a set of two distinct points. The output of the connected sum is then the manifold $X/\sim$, where $\sim$ is as above and

$$X = L\backslash(\overline{\varphi^{-1}(B(0,\epsilon))} \sqcup \overline{\psi^{-1}(B(0,\epsilon))}).$$

The advantage of this formulation is that p, q need not be in the same connected component: indeed we may perform the connected sum of any manifold L with itself along a pair of points.

Remark 1.11. The homeomorphism type of the connected sum of connected manifolds M, N is independent of the choices of p, q and φ, ψ, except that it may depend on the two possible orientations of the gluing map $\psi^{-1}\phi\varphi$. To prove this, one must appeal to the so-called *annulus theorem.*

Remark 1.12. By iterated connect sum of S^2 with T^2 and $\mathbb{R}P^2$, we can obtain all compact 2-dimensional manifolds.

Example 1.13. Let F be a topological space. A fiber bundle with fiber F is a triple (E, p, B), where E, B are topological spaces called the "total space" and "base", respectively, and $p : E \longrightarrow B$ is a continuous surjective map called the "projection map", such that, for each point $b \in B$, there is a neighbourhood U of b and a homeomorphism

$$\Phi : p^{-1}U \longrightarrow U \times F,$$

such that $p_U \circ \Phi = p$, where $p_U : U \times F \longrightarrow U$ is the usual projection. The submanifold $p^{-1}(b) \cong F$ is called the "fiber over b".

When B, F are topological manifolds, then clearly E becomes one as well. We will often encounter such manifolds.

Example 1.14 (General gluing construction)**.** To construct a topological manifold "from scratch", we glue open subsets of $\mathbb{R}^n$ together using homeomorphisms, as follows.

Begin with a countable collection of open subsets of $\mathbb{R}^n$: $\mathcal{A} = \{U_i\}$. Then for each i, we choose finitely many open subsets $U_{ij} \subset U_i$ and gluing maps

$$U_{ij} \xrightarrow{\varphi_{ij}} U_{ji} \ , \tag{9}$$

which we require to satisfy $\varphi_{ij}\varphi_{ji} = \mathrm{Id}_{U_{ji}}$, and such that $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all k, and most important of all, φ_{ij} must be *homeomorphisms.*

Next, we want the pairwise gluings to be consistent (transitive) and so we require that $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \mathrm{Id}_{U_{ij}\cap U_{jk}}$ for all i, j, k. This will ensure that the equivalence relation in (11) is well-defined.

Second countability of the glued manifold is guaranteed since we started with a countable collection of opens, but the Hausdorff property is not necessarily satisfied without a further assumption: we require that the graph of φ_{ij}, namely

$$\{(x, \varphi_{ij}(x)) \ : x \in U_{ij}\} \tag{10}$$

is a closed subset of $U_i \times U_j$.

The final glued topological manifold is then

$$M = \frac{\bigsqcup U_i}{\sim}, \tag{11}$$

for the equivalence relation $x \sim \varphi_{ij}(x)$ for $x \in U_{ij}$, for all i, j. This space has a distinguished atlas $\mathcal{A}$, whose charts are simply the inclusions of the U_i in $\mathbb{R}^n$.

Example 1.15 (Quotient construction)**.** Let Γ be a group, and give it the discrete topology. Suppose Γ acts continuously on the topological n-manifold M, meaning that the action map

$$\begin{aligned} \Gamma \times M &\xrightarrow{\rho} M \\ (h, x) &\longmapsto h \cdot x \end{aligned}$$

is continuous. Suppose also that the action is *free*, i.e. the stabilizer of each point is trivial. Suppose the action is *properly discontinuous*, meaning that each $x \in M$ has a neighbourhood U such that $h \cdot U$ is disjoint from U for all nontrivial $h \in \Gamma$, that is, for all $h \neq 1$. Finally, assume that the following subset is closed:

$$\{(x, y) \in M \times M \;:\; y = h \cdot x \text{ for some } h \in \Gamma\}$$

Then M/Γ is a topological manifold and $\pi : M \to M/\Gamma$ is a local homeomorphism.

Example 1.16 (Mapping torus)**.** Let M be a topological manifold and $\phi : M \to M$ a homeomorphism. Then

$$M_\phi = (M \times \mathbb{R})/\mathbb{Z}$$

is a manifold, where $k \in \mathbb{Z}$ acts via $k \cdot (p, t) = (\phi^k(p), t + k)$. This is called the mapping torus of ϕ and is a fibre bundle over $\mathbb{R}/\mathbb{Z} \cong S^1$ with fibre M.

Remark 1.17. In view of Example 1.14, it is natural to ask the following question: if $f : M \to N$ is a continuous map between topological manifolds M, N which are constructed by the general gluing construction, that is, if M is constructed from $\{U_i, U_{ij}, \varphi_{ij}\}$ and N from $\{V_i, V_{ij}, \psi_{ij}\}$, then how to describe the map f? First we need the collection of open subsets

$$U_i^k = f^{-1}(V_k), \qquad U_{ij}^k = U_i^k \cap U_{ij},$$

which satisfy the condition $\varphi_{ij}(U_{ij}^k) = U_{ji}^k$. Given this collection of opens, the map f determines and is determined by local continuous maps

$$f_i^k : U_i^k \to V_k$$

such that for all i, j, k, l, we have $\psi_{kl} \circ f_i^k = f_j^l \circ \varphi_{ij}$ on the open subset where the composition makes sense.