1.2 Smooth manifolds

Given coordinate charts (U_i, φ_i) and (U_j, φ_j) on a topological manifold, we can compare them along the intersection $U_{ij} = U_i \cap U_j$, by forming the "gluing map"

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij}). \tag{12}$$

This is a homeomorphism, since it is a composition of homeomorphisms. In this sense, topological manifolds are glued together by homeomorphisms.

This means that a given function on the manifold may happen to be differentiable in one chart but not in another, if the gluing map between the charts is not smooth – there is no way to make sense of calculus on topological manifolds. This is why we introduce smooth manifolds, where the gluing maps are *smooth*.

Remark 1.18 (Aside on smooth maps of vector spaces)**.** Let $U \subset V$ be an open set in a finite-dimensional vector space, and let $f : U \longrightarrow W$ be a function with values in another vector space W. We say f is differentiable at $p \in U$ if there is a linear map $Df(p) : V \longrightarrow W$ which approximates f near p, meaning that

$$\lim_{\substack{x \to 0 \\ x \neq 0}} \frac{||f(p+x) - f(p) - Df(p)(x)||}{||x||} = 0. \tag{13}$$

Notice that $Df(p)$ is uniquely characterized by the above property.

We have implicitly chosen inner products, and hence norms, on V and W in the above definition, though the differentiability of f is independent of this choice, since all norms are equivalent in finite dimensions. This is no longer true for infinite-dimensional vector spaces, where the norm or topology must be clearly specified and $Df(p)$ is required to be a continuous linear map. Most of what we do in this course can be developed in the setting of Banach spaces, i.e. complete normed vector spaces.

A basis for V has a corresponding dual basis $(x_1, \ldots, x_n)$ of linear functions on V, and we call these "coordinates". Similarly, let $(y_1, \ldots, y_m)$ be coordinates on W. Then the vector-valued function f has m scalar components $f_j = y_j \circ f$, and then the linear map $Df(p)$ may be written, relative to the chosen bases for V, W, as an $m \times n$ matrix, called the *Jacobian matrix* of f at p.

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{14}$$

We say that f is differentiable in U when it is differentiable at all $p \in U$, and we say it is continuously differentiable when

$$Df : U \longrightarrow \mathrm{Hom}(V, W) \tag{15}$$

is continuous. The vector space of continuously differentiable functions on U with values in W is called $C^1(U, W)$.

Notice that the first derivative Df is itself a map from U to a vector space $\mathrm{Hom}(V, W)$, so if its derivative exists, we obtain a map

$$D^2 f : U \longrightarrow \mathrm{Hom}(V, \mathrm{Hom}(V, W)), \tag{16}$$

and so on. The vector space of k times continuously differentiable functions on U with values in W is called $C^k(U, W)$. We are most interested in C^∞ or "smooth" maps, all of whose derivatives exist; the space of these is denoted $C^\infty(U, W)$, and so we have

$$C^\infty(U, W) = \bigcap_k C^k(U, W). \tag{17}$$

Note: for a C^2 function, $D^2 f$ actually has values in a smaller subspace of $V^* \otimes V^* \otimes W$, namely in $\mathrm{Sym}^2(V^*) \otimes W$, since "mixed partials are equal".

Definition 1.19. A *smooth manifold* is a topological manifold equipped with an equivalence class of smooth atlases, as explained next.

Definition 1.20. An atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ for a topological manifold is called *smooth* when all gluing maps

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij}) \tag{18}$$

are smooth maps, i.e. lie in $C^\infty(\varphi_i(U_{ij}), \mathbb{R}^n)$. Two atlases $\mathcal{A}, \mathcal{A}'$ are *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is itself a smooth atlas.

Remark 1.21. Instead of requiring an atlas to be smooth, we could ask for it to be C^k, or real-analytic, or even holomorphic (this makes sense for a $2n$-dimensional topological manifold when we identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$). This is how we define C^k, real-analytic, and complex manifolds, respectively.

We may now verify that all the examples from §1.1 are actually smooth manifolds:

Example 1.22 (Spheres)**.** The charts for the n-sphere given in Example 1.5 form a smooth atlas, since

$$\varphi_N \circ \varphi_S^{-1} : \vec{z} \mapsto \tfrac{1-x_0}{1+x_0}\vec{z} = \tfrac{(1-x_0)^2}{|\vec{x}|^2}\vec{z} = |\vec{z}|^{-2}\vec{z} \tag{19}$$

is a smooth map $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$, as required.

The Cartesian product of smooth manifolds inherits a natural smooth structure from taking the Cartesian product of smooth atlases. Hence the n-torus, for example, equipped with the atlas we described in Example 1.4, is smooth. Example 1.2 is clearly defining a smooth manifold, since the restriction of a smooth map to an open set is always smooth.

Example 1.23 (Projective spaces)**.** The charts for projective spaces given in Example 1.8 form a smooth atlas, since

$$\varphi_1 \circ \varphi_0^{-1}(z_1, \ldots, z_n) = (z_1^{-1}, z_1^{-1}z_2, \ldots, z_1^{-1}z_n), \tag{20}$$

which is smooth on $\mathbb{R}^n \backslash \{z_1 = 0\}$, as required, and similarly for all φ_i, φ_j.

The two remaining examples were constructed by gluing: the connected sum in Example 1.9 is clearly smooth since ϕ is a smooth map, and any topological manifold from Example 1.14 will be endowed with a natural smooth atlas as long as the gluing maps φ_{ij} are chosen to be C^∞.

1.3 Manifolds with boundary

Manifolds with boundary relate manifolds of different dimension. Since manifolds are not defined as subsets of another topological space, the notion of boundary is not the usual one from point set topology. To introduce boundaries, we change the local model for manifolds to

$$H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_n \geq 0\}, \tag{21}$$

with the induced topology from $\mathbb{R}^n$.

Definition 1.24. A topological manifold with boundary M is a second countable Hausdorff topological space which is locally homeomorphic to H^n. Its *boundary* ∂M is the $(n-1)$ manifold consisting of all points mapped to $x_n = 0$ by a chart, and its *interior* $\operatorname{Int} M$ is the set of points mapped to $x_n > 0$ by some chart. It follows that $M = \partial M \sqcup \operatorname{Int} M$.

A smooth structure on such a manifold *with boundary* is an equivalence class of smooth atlases, with smoothness as defined below.

Definition 1.25. Let V, W be finite-dimensional vector spaces, as before. A function $f : A \longrightarrow W$ from an arbitrary subset $A \subset V$ is smooth when it admits a smooth extension to an open neighbourhood $U_p \subset W$ of every point $p \in A$.

Example 1.26. The function $f(x, y) = y$ is smooth on H^2 but $f(x, y) = \sqrt{y}$ is not, since its derivatives do not extend to $y \leq 0$.

Remark 1.27. If M is an n-manifold with boundary, then $\operatorname{Int} M$ is a usual n-manifold (without boundary). Also, ∂M is an $n-1$-manifold without boundary. This is sometimes phrased as the equation

$$\partial^2 = 0. \tag{22}$$

Example 1.28 (Möbius strip)**.** Consider the quotient of $\mathbb{R} \times [0, 1]$ by the identification $(x, y) \sim (x + 1, 1 - y)$. The result E is a manifold with boundary. It is also a fiber bundle over S^1, via the map $\pi : [(x, y)] \mapsto e^{2\pi i x}$. The boundary, ∂E, is isomorphic to S^1, so this provides us with our first example of a non-trivial fiber bundle, since the trivial fiber bundle $S^1 \times [0, 1]$ has disconnected boundary.

1.4 Cobordism

Compact $(n+1)$-Manifolds with boundary provide us with a natural equivalence relation on compact n-manifolds, called *cobordism*.

Definition 1.29. Compact n-manifolds M_1, M_2 are *cobordant* when there exists N, a compact $n+1$-manifold with boundary, such that ∂N is isomorphic to the disjoint union $M_1 \sqcup M_2$. All manifolds cobordant to M form the *cobordism class* of M. We say that M is null-cobordant if $M = \partial N$ for N a compact $n + 1$–manifold with boundary.

Remark 1.30. It is important to assume compactness, otherwise all manifolds are null-cobordant, by taking Cartesian product with the noncompact manifold with boundary $[0, 1)$.

Let Ω^n be the set of cobordism classes of compact n-manifolds, including the empty set $\varnothing$ as a compact n-manifold. Using the disjoint union operation $[M_1] + [M_2] = [M_1 \sqcup M_2]$, we see that Ω^n is an abelian group with identity $[\varnothing]$.

The direct sum $\Omega^\bullet = \oplus_{n \geq 0} \Omega^n$ is then endowed with another operation,

$$[M_1] \cdot [M_2] = [M_1 \times M_2], \tag{23}$$

rendering $\Omega^\bullet$ into a commutative ring, called the *cobordism ring*. It has a multiplicative unit $[*]$, the class of the 0-manifold consisting of a single point. It is also graded by dimension.

Proposition 1.31. *The cobordism ring is 2-torsion, i.e.* $x + x = 0 \quad \forall x$.

Proof. For any manifold M, the manifold with boundary $M \times [0,1]$ has boundary $M \sqcup M$. Hence $[M] + [M] = [\varnothing] = 0$, as required. ☐

Example 1.32. The n-sphere S^n is null-cobordant (i.e. cobordant to $\varnothing$), since $\partial B_{n+1}(0,1) \cong S^n$, where $B_{n+1}(0,1)$ denotes the unit ball in $\mathbb{R}^{n+1}$.

Example 1.33. Any oriented compact 2-manifold is null-cobordant: we may embed it in $\mathbb{R}^3$ and the "inside" is a 3-manifold with boundary.

We now state an amazing theorem of Thom, which is a complete description of the cobordism ring of smooth compact n-manifolds.

Theorem 1.34. *The cobordism ring is a (countably generated) polynomial ring over* $\mathbb{F}_2$ *with generators in every dimension* $n \neq 2^k - 1$, *i.e.*

$$\Omega^\bullet = \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \ldots]. \tag{24}$$

This theorem implies that there are 3 cobordism classes in dimension 4, namely x_2^2, x_4, and $x_2^2 + x_4$. Can you find 4-manifolds representing these classes? Can you find *connected* representatives?

Remark 1.35. Dold constructed the family of manifolds

$$P(m,n) = (S^m \times \mathbb{C}P^n)/((x,y) \sim (-x, \bar{y})),$$

and showed that the generator x_n above is represented by the manifold $P(2^r, s2^r)$, where $n = 2^r(2s+1)$.