2.2 The derivative

A description of the tangent bundle is not complete without defining the derivative of a general smooth map of manifolds $f : M \longrightarrow N$. Such a map may be defined locally in charts (U_i, φ_i) for M and (V_α, ψ_α) for N as a collection of vector-valued functions $\psi_\alpha \circ f \circ \varphi_i^{-1} = f_{i\alpha} : \varphi_i(U_i) \longrightarrow \psi_\alpha(V_\alpha)$ which satisfy

$$(\psi_\beta \circ \psi_\alpha^{-1}) \circ f_{i\alpha} = f_{j\beta} \circ (\varphi_j \circ \varphi_i^{-1}). \tag{35}$$

Differentiating, we obtain

$$D(\psi_\beta \circ \psi_\alpha^{-1}) \circ Df_{i\alpha} = Df_{j\beta} \circ D(\varphi_j \circ \varphi_i^{-1}). \tag{36}$$

Equation 36 shows that $Df_{i\alpha}$ and $Df_{j\beta}$ glue together to define a map $TM \longrightarrow TN$. This map is called the derivative of f and is denoted $Df : TM \longrightarrow TN$. Sometimes it is called the "push-forward" of vectors and is denoted f_*. The map fits into the commutative diagram

$$\begin{tikzcd} TM \arrow[r, "Df"] \arrow[d, "\pi"'] & TN \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & N \end{tikzcd} \tag{37}$$

Each fiber $\pi^{-1}(x) = T_xM \subset TM$ is a vector space, and the map $Df : T_xM \longrightarrow T_{f(x)}N$ is a linear map. In fact, (f, Df) defines a homomorphism of vector bundles from TM to TN.

The usual chain rule for derivatives then implies that if $f \circ g = h$ as maps of manifolds, then $Df \circ Dg = Dh$. As a result, we obtain the following category-theoretic statement.

Proposition 2.4. *The mapping T which assigns to a manifold M its tangent bundle TM, and which assigns to a map $f : M \longrightarrow N$ its derivative $Df : TM \longrightarrow TN$, is a functor from the category of manifolds and smooth maps to itself*[1].

For this reason, the derivative map Df is sometimes called the "tangent mapping" Tf.

2.3 Vector fields

A vector field on an open subset $U \subset V$ of a vector space V is what we usually call a vector-valued function, i.e. a function $X : U \to V$. If $(x_1, \ldots, x_n)$ is a basis for V^*, hence a coordinate system for V, then the constant vector fields dual to this basis are usually denoted in the following way:

$$\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right). \tag{38}$$

The reason for this notation is that we may identify a vector v with the operator of directional derivative in the direction v. We will see later that

[1]We can also say that it is a functor from manifolds to the category of smooth vector bundles.

vector fields may be viewed as derivations on functions. A derivation is a linear map D from smooth functions to $\mathbb{R}$ satisfying the Leibniz rule $D(fg) = fDg + gDf$.

The tangent bundle allows us to make sense of the notion of vector field in a global way. Locally, in a chart (U_i, φ_i), we would say that a vector field X_i is simply a vector-valued function on U_i, i.e. a function $X_i : \varphi(U_i) \longrightarrow \mathbb{R}^n$. Of course if we had another vector field X_j on (U_j, φ_j), then the two would agree as vector fields on the overlap $U_i \cap U_j$ when $D(\varphi_j \circ \varphi_i^{-1}) : X_i \mapsto X_j$. So, if we specify a collection $\{X_i \in C^\infty(U_i, \mathbb{R}^n)\}$ which glue together on overlaps, it defines a global vector field.

Definition 2.5. A smooth vector field on the manifold M is a smooth map $X : M \longrightarrow TM$ such that $\pi \circ X = \mathrm{id}_M$. In words, it is a smooth assignment of a unique tangent vector to each point in M.

Such maps X are also called *cross-sections* or simply *sections* of the tangent bundle TM, and the set of all such sections is denoted $C^\infty(M, TM)$ or, better, $\Gamma^\infty(M, TM)$, to distinguish them from all smooth maps $M \longrightarrow TM$. The space vector fields is also sometimes denoted by $\mathfrak{X}(M)$.

Example 2.6. From a computational point of view, given an atlas $(\tilde{U}_i, \varphi_i)$ for M, let $U_i = \varphi_i(\tilde{U}_i) \subset \mathbb{R}^n$ and let $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$. Then a global vector field $X \in \Gamma^\infty(M, TM)$ is specified by a collection of vector-valued functions

$$X_i : U_i \longrightarrow \mathbb{R}^n, \tag{39}$$

such that

$$D\varphi_{ij}(X_i(x)) = X_j(\varphi_{ij}(x)) \tag{40}$$

for all $x \in \varphi_i(\tilde{U}_i \cap \tilde{U}_j)$. For example, if $S^1 = U_0 \sqcup U_1 / \sim$, with $U_0 = \mathbb{R}$ and $U_1 = \mathbb{R}$, with $x \in U_0 \backslash \{0\} \sim y \in U_1 \backslash \{0\}$ whenever $y = x^{-1}$, then $\varphi_{01} : x \mapsto x^{-1}$ and $D\varphi_{01}(x) : v \mapsto -x^{-2}v$. Then if we define (letting x be the standard coordinate along $\mathbb{R}$)

$$\begin{aligned} X_0 &= \frac{\partial}{\partial x} \\ X_1 &= -y^2 \frac{\partial}{\partial y}, \end{aligned}$$

we see that this defines a global vector field, which does not vanish in U_0 but vanishes to order 2 at a single point in U_1. Find the local expression in these charts for the rotational vector field on S^1 given in polar coordinates by $\frac{\partial}{\partial \theta}$.

Remark 2.7. While a vector $v \in T_pM$ is mapped to a vector $(Df)_p(v) \in T_{f(p)}N$ by the derivative of a map $f \in C^\infty(M, N)$, there is no way, in general, to transport a vector field X on M to a vector field on N. If f is invertible, then of course $Df \circ X \circ f^{-1} : N \to TN$ defines a vector field on N, which can be called f_*X, but if f is not invertible this approach fails.

Definition 2.8. We say that $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are f–related, for $f \in C^\infty(M,N)$, when the following diagram commutes

$$\begin{tikzcd} TM \arrow[r, "Df"] & TN \\ M \arrow[u, "X"] \arrow[r, "f"'] & N \arrow[u, "Y"'] \end{tikzcd} . \tag{41}$$

2.4 Local structure of smooth maps

In some ways, smooth manifolds are easier to produce or find than general topological manifolds, because of the fact that smooth maps have linear approximations. Therefore smooth maps often behave like linear maps of vector spaces, and we may gain inspiration from vector space constructions (e.g. subspace, kernel, image, cokernel) to produce new examples of manifolds.

In charts (U, φ), (V, ψ) for the smooth manifolds M, N, a smooth map $f : M \longrightarrow N$ is represented by a smooth map $\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$. We shall give a general local classification of such maps, based on the behaviour of the derivative. The fundamental result which provides information about the map based on its derivative is the *inverse function theorem.*

Theorem 2.9 (Inverse function theorem)**.** *Let $f : (M,p) \to (N,q)$ be a smooth map of n-dimensional manifolds and suppose that $Df(p) : T_pM \to T_qN$ is invertible. Then f has a local smooth inverse. That is, there are neighbourhoods U, V of p, q and a smooth map $g : V \to U$ such that $f \circ g = \mathrm{id}_V$ and $g \circ f = \mathrm{id}_U$.*

Proof. Without loss of generality, we can take M to be a neighbourhood of the origin in $\mathbb{R}^n$ and $N = \mathbb{R}^n$, and assume that $f(0) = 0$. We can also assume $Df(p) = \mathrm{Id}$, since we can replace f by $(Df(0))^{-1} \circ f$ (linear change of variables). We are trying to invert f, so solve the equation $y = f(x)$ uniquely for x. Define k so that $f(x) = x + k(x)$. Hence $k(x)$ is the nonlinear part of f.

The claim is that if y is in a sufficiently small neighbourhood of the origin, then the map $h_y : x \mapsto y - k(x)$ is a contraction mapping on some closed ball; it then has a unique fixed point $g(y)$, and so $y - k(g(y)) = g(y)$, i.e. g is an inverse for f.

Why is h_y a contraction mapping? Note that $Dh_y(0) = 0$ and hence there is a ball $B(0,r)$ where $||Dh_y|| \le \frac{1}{2}$. This then implies (mean value theorem) that for $x, x' \in B(0,r)$,

$$||h_y(x) - h_y(x')|| \le \tfrac{1}{2}||x - x'||.$$

Therefore h_y does look like a contraction, we just have to make sure it's operating on a complete metric space. Let's estimate the size of $h_y(x)$:

$$||h_y(x)|| \le ||h_y(x) - h_y(0)|| + ||h_y(0)|| \le \tfrac{1}{2}||x|| + ||y||.$$

Therefore by taking $y \in B(0, \frac{r}{2})$, the map h_y is a contraction mapping on $\overline{B(0,r)}$. Let $g(y)$ be the unique fixed point of h_y guaranteed by the contraction mapping theorem.

To see that ϕ is continuous (and hence f is a homeomorphism), we compute

$$\begin{aligned} ||g(y) - g(y')|| &= ||h_y(g(y)) - h_{y'}(g(y'))|| \\ &\leq ||h_y(g(y)) - h_y(g(y'))|| + ||y - y'|| \\ &\leq \tfrac{1}{2}||g(y) - g(y')|| + ||y - y'||, \end{aligned}$$

so that we have $||g(y) - g(y')|| \leq 2||y - y'||$, as required.

Having shown that g is continuous, we can choose an open set $U \subset B(0, r)$ and define $V = g^{-1}(U) \subset B(0, \frac{r}{2})$. Then $f \circ g = \mathrm{id}_V$ by the fixed point property and $g \circ f = \mathrm{id}_U$ by the uniqueness of fixed points in the closed ball, proving that $f : U \to V$ is indeed a homeomorphism.

To see that g is differentiable, we guess the derivative $(Df)^{-1}$ and compute. Let $x = g(y)$ and $x' = g(y')$. For this to make sense we must have chosen r small enough so that Df is nonsingular on $\overline{B(0, r)}$, which is not a problem.

$$\begin{aligned} ||g(y) - g(y') - (Df(x))^{-1}(y - y')|| &= ||x - x' - (Df(x))^{-1}(f(x) - f(x'))|| \\ &\leq ||(Df(x))^{-1}||||(Df(x))(x - x') - (f(x) - f(x'))||. \end{aligned}$$

Now note that $||(Df(x))^{-1}||$ is bounded and $||x - x'|| \leq 2||y - y'||$ as shown before. Dividing by $||y - y'||$, taking the limit $y \to y'$, and using the differentiability of f, we get that g is differentiable, and with derivative $(Df)^{-1}$. That is,

$$Dg = (Df)^{-1}. \tag{42}$$

Since inversion is C^∞, g has as many derivatives as f, hence g is C^∞. $\square$

This theorem provides us with a local normal form for a smooth map with $Df(p)$ invertible: we may choose coordinates on sufficiently small neighbourhoods of $p, f(p)$ so that f is represented by the identity map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

In fact, the inverse function theorem leads to a normal form theorem for a more general class of maps:

Theorem 2.10 (Constant rank theorem)**.** *Let $f : M^m \to N^n$ be a smooth map such that Df has constant rank k in a neighbourhood of $p \in M$. Then there are charts (U, φ) and (V, ψ) containing $p, f(p)$ such that*

$$\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0). \tag{43}$$

Proof. Begin by choosing charts so that without loss of generality M is an open set in $\mathbb{R}^m$ and N is $\mathbb{R}^n$.

Since rk $Df = k$ at p, there is a $k \times k$ minor of $Df(p)$ with nonzero determinant. Reorder the coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$ so that this minor is top left, and translate coordinates so that $f(0) = 0$. label the coordinates $(x_1, \ldots, x_k, y_1, \ldots y_{m-k})$ on the domain and $(u_1, \ldots u_k, v_1, \ldots, v_{n-k})$ on the codomain.

Then we may write $f(x, y) = (Q(x, y), R(x, y))$, where Q is the projection to $u = (u_1, \ldots, u_k)$ and R is the projection to v. with $\frac{\partial Q}{\partial x}$ nonsingular. First we wish to put Q into normal form. Consider the map

$\phi(x, y) = (Q(x, y), y)$, which has derivative

$$D\phi = \begin{pmatrix} \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \\ 0 & 1 \end{pmatrix} \quad (44)$$

As a result we see $D\phi(0)$ is nonsingular and hence there exists a local inverse $\phi^{-1}(x, y) = (A(x, y), B(x, y))$. Since it's an inverse this means $(x, y) = \phi(\phi^{-1}(x, y)) = (Q(A, B), B)$, which implies that $B(x, y) = y$.

Then $f \circ \phi^{-1} : (x, y) \mapsto (x, S = R(A, y))$, and must still be of rank k. Since its derivative is

$$D(f \circ \phi^{-1}) = \begin{pmatrix} I_{k\times k} & 0 \\ \frac{\partial S}{\partial x} & \frac{\partial S}{\partial y} \end{pmatrix} \quad (45)$$

we conclude that $\frac{\partial S}{\partial y} = 0$, meaning that we have eliminated the y-dependence:

$$f \circ \phi^{-1} : (x, y) \mapsto (x, S(x)). \quad (46)$$

We now postcompose by the diffeomorphism $\sigma : (u, v) \mapsto (u, v - S(u))$, to obtain

$$\sigma \circ f \circ \phi^{-1} : (x, y) \mapsto (x, 0), \quad (47)$$

as required. □

As we shall see, these theorems have many uses. One of the most straightforward uses is for defining submanifolds.

There are several ways to define the notion of submanifold. We will use a definition which works for topological and smooth manifolds, based on the local model of inclusion of a vector subspace. These are sometimes called *regular* or *embedded* submanifolds.

Definition 2.11. A subspace $L \subset M$ of an m-manifold is called a submanifold of codimension k when each point $x \in L$ is contained in a chart (U, φ) for M such that

$$L \cap U = f^{-1}(0), \quad (48)$$

where f is the composition of φ with the projection $\mathbb{R}^m \to \mathbb{R}^k$ to the last k coordinates $(x_{m-k+1}, \ldots, x_m)$. A submanifold of codimension 1 is usually called a *hypersurface.*

Proposition 2.12. *If $f : M \longrightarrow N$ is a smooth map of manifolds, and if $Df(p)$ has constant rank on M, then for any $q \in f(M)$, the inverse image $f^{-1}(q) \subset M$ is a regular submanifold.*

Proof. Let $x \in f^{-1}(q)$. Then there exist charts ψ, φ such that $\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$ and $f^{-1}(q) \cap U = \{x_1 = \cdots = x_k = 0\}$. Hence we obtain that $f^{-1}(q)$ is a codimension k submanifold. □

Example 2.13. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by $(x_1, \ldots, x_n) \mapsto \sum x_i^2$. Then $Df(x) = (2x_1, \ldots, 2x_n)$, which has rank 1 at all points in $\mathbb{R}^n \backslash \{0\}$. Hence since $f^{-1}(q)$ contains $\{0\}$ iff $q = 0$, we see that $f^{-1}(q)$ is a regular submanifold for all $q \neq 0$. Exercise: show that this manifold structure is compatible with that obtained in Example 1.22.

The previous example leads to the following special case.

Proposition 2.14. *If $f : M \longrightarrow N$ is a smooth map of manifolds and $Df(p)$ has rank equal to $\dim N$ along $f^{-1}(q)$, then this subset $f^{-1}(q)$ is an embedded submanifold of M.*

Proof. Since the rank is maximal along $f^{-1}(q)$, it must be maximal in an open neighbourhood $U \subset M$ containing $f^{-1}(q)$, and hence $f : U \longrightarrow N$ is of constant rank. □

Definition 2.15. If $f : M \longrightarrow N$ is a smooth map such that $Df(p)$ is surjective, then p is called a *regular point*. Otherwise p is called a *critical point*. If all points in the level set $f^{-1}(q)$ are regular points, then q is called a *regular value*, otherwise q is called a critical value. In particular, if $f^{-1}(q) = \varnothing$, then q is regular.

It is often useful to highlight two classes of smooth maps; those for which Df is everywhere *injective*, or, on the other hand *surjective*.

Definition 2.16. A smooth map $f : M \longrightarrow N$ is called a *submersion* when $Df(p)$ is surjective at all points $p \in M$, and is called an *immersion* when $Df(p)$ is injective at all points $p \in M$. If f is an injective immersion which is a homeomorphism onto its image (when the image is equipped with subspace topology), then we call f an *embedding*.

Proposition 2.17. *If $f : M \longrightarrow N$ is an embedding, then $f(M)$ is a regular submanifold.*

Proof. Let $f : M \longrightarrow N$ be an embedding. Then for all $m \in M$, we have charts (U, φ), (V, ψ) where $\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$. If $f(U) = f(M) \cap V$, we're done. To make sure that some other piece of M doesn't get sent into the neighbourhood, use the fact that $f(U)$ is open in the subspace topology. This means we can find a smaller open set $V' \subset V$ such that $V' \cap f(M) = f(U)$. Restricting the coordinates to V', we see that $f(M)$ is cut out by $(x_{m+1}, \ldots, x_n)$, where $n = \dim N$. □

Example 2.18. If $\iota : M \longrightarrow N$ is an embedding of M into N, then $D\iota : TM \longrightarrow TN$ is also an embedding (hence so are $D^k\iota : T^kM \longrightarrow T^kN$), showing that TM is a submanifold of TN.

2.5 Smooth maps between manifolds with boundary

We may also use the constant rank theorem to study manifolds with boundary.

Proposition 2.19. *Let M be a smooth n-manifold and $f : M \longrightarrow \mathbb{R}$ a smooth and proper real-valued function, and let a, b, with $a < b$, be regular values of f. Then $f^{-1}([a, b])$ is a cobordism between the closed $n-1$-manifolds $f^{-1}(a)$ and $f^{-1}(b)$.*

Proof. The pre-image $f^{-1}((a, b))$ is an open subset of M and hence a submanifold. Since p is regular for all $p \in f^{-1}(a)$, we may (by the constant rank theorem) find charts such that f is given near p by the linear map

$$(x_1, \ldots, x_m) \mapsto x_m. \tag{49}$$

Possibly replacing x_m by $-x_m$, we therefore obtain a chart near p for $f^{-1}([a,b])$ into H^m, as required. Proceed similarly for $p \in f^{-1}(b)$. $\square$

Example 2.20. Using $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $(x_1, \dots, x_n) \mapsto \sum x_i^2$, this gives a simple proof for the fact that the closed unit ball $\overline{B(0,1)} = f^{-1}([-1,1])$ is a manifold with boundary.

Example 2.21. Consider the C^∞ function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $(x,y,z) \mapsto x^2 + y^2 - z^2$. Both $+1$ and -1 are regular values for this map, with pre-images given by 1- and 2-sheeted hyperboloids, respectively. Hence $f^{-1}([-1,1])$ is a cobordism between hyperboloids of 1 and 2 sheets. In other words, it defines a cobordism between the disjoint union of two closed disks and the closed cylinder (each of which has boundary $S^1 \sqcup S^1$). Does this cobordism tell us something about the cobordism class of a connected sum?

Proposition 2.22. *Let $f : M \longrightarrow N$ be a smooth map from a manifold with boundary to the manifold N. Suppose that $q \in N$ is a regular value of f and also of $f|_{\partial M}$. Then the pre-image $f^{-1}(q)$ is a submanifold with boundary*[2]*. Furthermore, the boundary of $f^{-1}(q)$ is simply its intersection with ∂M.*

Proof. If $p \in f^{-1}(q)$ is not in ∂M, then as before $f^{-1}(q)$ is a submanifold in a neighbourhood of p. Therefore suppose $p \in \partial M \cap f^{-1}(q)$. Pick charts φ, ψ so that $\varphi(p) = 0$ and $\psi(q) = 0$, and $\psi f \varphi^{-1}$ is a map $U \subset H^m \longrightarrow \mathbb{R}^n$. Extend this to a smooth function $\tilde{f}$ defined in an open set $\tilde{U} \subset \mathbb{R}^m$ containing U. Shrinking $\tilde{U}$ if necessary, we may assume $\tilde{f}$ is regular on $\tilde{U}$. Hence $\tilde{f}^{-1}(0)$ is a submanifold of $\mathbb{R}^m$ of codimension n.

Now consider the real-valued function $\pi : \tilde{f}^{-1}(0) \longrightarrow \mathbb{R}$ given by the restriction of $(x_1, \dots, x_m) \mapsto x_m$. $0 \in \mathbb{R}$ must be a regular value of π, since if not, then the tangent space to $\tilde{f}^{-1}(0)$ at 0 would lie completely in $x_m = 0$, which contradicts the fact that q is a regular point for $f|_{\partial M}$.

Hence, by Proposition 2.19, we have expressed $f^{-1}(q)$, in a neighbourhood of p, as a regular submanifold with boundary given by $\{\varphi^{-1}(x) : x \in \tilde{f}^{-1}(0) \text{ and } \pi(x) \geq 0\}$, as required. $\square$

[2] i.e. locally modeled on the inclusion $H^k \subset H^n$ given by $(x_1, \dots x_k) \mapsto (0, \dots, 0, x_1, \dots x_k)$.