3 Transversality

We continue to use the constant rank theorem to produce more manifolds, except now these will be cut out only *locally* by functions. Globally, they are cut out by intersecting with another submanifold. You should think that intersecting with a submanifold locally imposes a number of constraints equal to its codimension.

The problem is that the intersection of submanifolds need not be a submanifold; this is why the condition of transversality is so important - it guarantees that intersections are smooth.

Two subspaces $K, L \subset V$ of a vector space V are *transverse* when $K + L = V$, i.e. every vector in V may be written as a (possibly non-unique) linear combination of vectors in K and L. In this situation one can easily see that $\dim V = \dim K + \dim L - \dim K \cap L$, or equivalently

$$\mathrm{codim}(K \cap L) = \mathrm{codim} K + \mathrm{codim} L. \tag{50}$$

We may apply this to submanifolds as follows:

Definition 3.1. Let $K, L \subset M$ be regular submanifolds such that every point $p \in K \cap L$ satisfies

$$T_pK + T_pL = T_pM. \tag{51}$$

Then K, L are said to be *transverse* submanifolds and we write $K \pitchfork L$.

Proposition 3.2. *If $K, L \subset M$ are transverse submanifolds, then $K \cap L$ is either empty, or a submanifold of codimension* $\mathrm{codim}K + \mathrm{codim}L$.

Proof. Let $p \in K \cap L$. Then there is a neighbourhood U of p for which $K \cap U = f^{-1}(0)$ for 0 a regular value of a function $f : U \longrightarrow \mathbb{R}^{\mathrm{codim}K}$ and $L \cap U = g^{-1}(0)$ for 0 a regular value of a function $g : L \cap U \longrightarrow \mathbb{R}^{\mathrm{codim}L}$.

Then p must be a regular point for $(f, g) : L \cap M \cap U \longrightarrow \mathbb{R}^{\mathrm{codim}K+\mathrm{codim}L}$, since the kernel of its derivative is the intersection $\ker Df(p) \cap \ker Dg(p)$, which is exactly $T_pK \cap T_pL$, which has codimension $\mathrm{codim}K + \mathrm{codim}L$ by the transversality assumption, implying $D(f, g)(p)$ is surjective. Therefore $(f, g)|_{\tilde{U}}^{-1}(0, 0) = f^{-1}(0) \cap g^{-1}(0) = K \cap L \cap \tilde{U}$ is a submanifold. □

Example 3.3 (Exotic spheres)**.** Consider the following intersections in $\mathbb{C}^5 \backslash 0$:

$$S^7_k = \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}. \tag{52}$$

This is a transverse intersection, and for $k = 1, \ldots, 28$ the intersection is a smooth manifold homeomorphic to S^7. These exotic 7-spheres were constructed by Brieskorn and represent each of the 28 diffeomorphism classes on S^7.

We may choose to phrase the previous transversality result in a slightly different way, in terms of the embedding maps k, l for K, L in M. Specifically, we say the maps k, l are transverse in the sense that $\forall a \in K, b \in L$ such that $k(a) = l(b) = p$, we have $\mathrm{im}(Dk(a)) + \mathrm{im}(Dl(b)) = T_pM$. The advantage of this approach is that it makes sense for any maps, not necessarily embeddings.

Definition 3.4. Two maps $f : K \longrightarrow M$, $g : L \longrightarrow M$ of manifolds are called *transverse* when $\mathrm{im}(Df(a)) + \mathrm{im}(Dg(b)) = T_pM$ for all a, b, p such that $f(a) = g(b) = p$.

Proposition 3.5. *If $f : K \longrightarrow M$, $g : L \longrightarrow M$ are transverse smooth maps, then $K{}_f\times_g L = \{(a, b) \in K \times L \ : \ f(a) = g(b)\}$ is naturally a smooth manifold equipped with commuting maps*

$$\begin{tikzcd} K \times L \arrow[drr, "p_2"] \arrow[ddr, "p_1"'] & & \\ & K{}_f\times_g L \arrow[ul, "i"'] \arrow[r] \arrow[d] \arrow[dr, "f\cap g"] & L \arrow[d, "g"] \\ & K \arrow[r, "f"'] & M \end{tikzcd} \tag{53}$$

where i is the inclusion and $f \cap g : (a, b) \mapsto f(a) = g(b)$.

The manifold $K{}_f\times_g L$ of the previous proposition is called the *fiber product* of K with L over M, and is a generalization of the intersection product. It is often denoted simply by $K \times_M L$, when the maps to M are clear.

Proof. Consider the graphs $\Gamma_f \subset K \times M$ and $\Gamma_g \subset L \times M$. To impose $f(k) = g(l)$, we can take an intersection with the diagonal submanifold

$$\Delta = \{(k, m, l, m) \in K \times M \times L \times M\}. \tag{54}$$

Step 1. We show that the intersection $\Gamma = (\Gamma_f \times \Gamma_g) \cap \Delta$ is transverse. Let $f(k) = g(l) = m$ so that $x = (k, m, l, m) \in \Gamma$, and note that

$$T_x(\Gamma_f \times \Gamma_g) = \{((v, Df(v)), (w, Dg(w))), \ v \in T_kK, \quad w \in T_lL\} \tag{55}$$

whereas we also have

$$T_x(\Delta) = \{((v, m), (w, m)) \ : \ v \in T_kK, \quad w \in T_lL, \quad m \in T_pM\} \tag{56}$$

By transversality of f, g, any tangent vector $m_i \in T_pM$ may be written as $Df(v_i) + Dg(w_i)$ for some (v_i, w_i), $i = 1, 2$. In particular, we may decompose a general tangent vector to $M \times M$ as

$$(m_1, m_2) = (Df(v_2), Df(v_2)) + (Dg(w_1), Dg(w_1)) + (Df(v_1 - v_2), Dg(w_2 - w_1)), \tag{57}$$

leading directly to the transversality of the spaces (55), (56). This shows that Γ is a submanifold of $K \times M \times L \times M$.
Step 2. The projection map $\pi : K \times M \times L \times M \to K \times L$ takes Γ bijectively to $K{}_f\times_g L$. Since (55) is a graph, it follows that $\pi|_\Gamma : \Gamma \to K \times L$ is an injective immersion. Since the projection π is an open map, it also follows that $\pi|_\Gamma$ is a homeomorphism onto its image, hence is an embedding. This shows that $K{}_f\times_g L$ is a submanifold of $K \times L$. □

Example 3.6. If $K_1 = M \times Z_1$ and $K_2 = M \times Z_2$, we may view both K_i as "fibering" over M with fibers Z_i. If p_i are the projections to M, then $K_1 \times_M K_2 = M \times Z_1 \times Z_2$, hence the name "fiber product".

Example 3.7. Consider the Hopf map $p : S^3 \longrightarrow S^2$ given by composing the embedding $S^3 \subset \mathbb{C}^2 \backslash \{0\}$ with the projection $\pi : \mathbb{C}^2 \backslash \{0\} \longrightarrow \mathbb{C}P^1 \cong S^2$. Then for any point $q \in S^2$, $p^{-1}(q) \cong S^1$. Since p is a submersion, it is obviously transverse to itself, hence we may form the fiber product

$$S^3 \times_{S^2} S^3,$$

which is a smooth 4-manifold equipped with a map $p \cap p$ to S^2 with fibers $(p \cap p)^{-1}(q) \cong S^1 \times S^1$.

These are our first examples of nontrivial fiber bundles, which we shall explore later.

The following result is an exercise: just as we may take the product of a manifold with boundary K with a manifold without boundary L to obtain a manifold with boundary $K \times L$, we have a similar result for fiber products.

Proposition 3.8. *Let K be a manifold with boundary where L, M are without boundary. Assume that $f : K \longrightarrow M$ and $g : L \longrightarrow M$ are smooth maps such that both f and ∂f are transverse to g. Then the fiber product $K \times_M L$ is a manifold with boundary equal to $\partial K \times_M L$.*

3.1 Stability

Transversality is a stable condition. In other words, if transversality holds, it will continue to hold for any sufficiently small perturbation (of the submanifolds or maps involved). Not only is transversality *stable*, it is actually *generic*, meaning that even if it does not hold, it can be made to hold by a small perturbation. In a sense, stability says that transversal maps form an open set, and genericity says that this open set is dense in the space of maps. To make this precise, we would introduce a topology on the space of maps, something which we leave for another course.

Definition 3.9. We call a smooth map

$$F : M \times [0, 1] \to N \tag{58}$$

a smooth homotopy from f_0 to f_1, where $f_t = F \circ j_t$ and $j_t : M \to M \times [0, 1]$ is the embedding $x \mapsto (x, t)$.

Definition 3.10. A property of a smooth map $f : M \longrightarrow N$ is *stable* under perturbations when for any smooth homotopy f_t with $f_0 = f$, there exists an $\epsilon > 0$ such that the property holds for all f_t with $t < \epsilon$.

Proposition 3.11. *If M is compact, then the property of $f : M \to N$ being an immersion (or submersion) is stable under perturbations.*

Proof. If $f_t, t \in [0, 1]$ is a smooth homotopy of the immersion f_0, then in any chart around the point $p \in M$, the derivative $Df_0(p)$ has a $m \times m$ submatrix with nonvanishing determinant, for $m = \dim M$. By continuity, this $m \times m$ submatrix must have nonvanishing determinant in a neighbourhood around $(p, 0) \in M \times [0, 1]$. We can cover $M \times \{0\}$ by a finite number of such neighbourhoods, since M is compact. Choose ϵ such that $M \times [0, \epsilon)$ is contained in the union of these intervals, giving the result. The proof for submersions is identical. $\square$

Corollary 3.12. *If K is compact and $f : K \to M$ is transverse to the closed submanifold $L \subset M$ (this just means that f is transverse to the embedding $\iota : L \to M$), then the transversality is stable under perturbations of f.*

Proof. Let $F : K \times [0,1] \to M$ be a homotopy with $f_0 = f$. We show that K has an open cover by neighbourhoods in which f_t is transverse for t in a small interval; we then use compactness to obtain a uniform interval.

First the points which do not intersect L: $F^{-1}(M \setminus L)$ is open in $K \times [0,1]$ and contains $(K \setminus f^{-1}(L)) \times \{0\}$. So, for each $p \in K \setminus f^{-1}(L)$, there is a neighbourhood $U_p \subset K$ of p and an interval $I_p = [0, \epsilon_p)$ such that $F(U_p \times I_p) \cap L = \varnothing$.

Now, the points which do intersect L: L is a submanifold, so for each $p \in f^{-1}(L)$, we can find a neighbourhood $V \subset M$ containing $f(p)$ and a submersion $\psi : V \to \mathbb{R}^l$ cutting out $L \cap V$. Transversality of f and L is then the statement that ψf is a submersion at p. This implies there is a neighbourhood $\tilde{U}_p$ of $(p, 0)$ in $K \times [0,1]$ where ψf_t is a submersion. Choose an open subset (containing $(p, 0)$) of the form $U_p \times I_p$, for $I_p = [0, \epsilon_p)$.

By compactness of K, choose a finite subcover of $\{U_p\}_{p \in K}$; the smallest ϵ_p in the resulting subcover gives the required interval in which f_t remains transverse to L. □

Remark 3.13. Transversality of two maps $f : M \to N$, $g : M' \to N$ can be expressed in terms of the transversality of $f \times g : M \times M' \to N \times N$ to the diagonal $\Delta_N \subset N \times N$. So, if M and M' are compact, we get stability for transversality of f, g under perturbations of both f and g.

Remark 3.14. Local diffeomorphism and embedding are also stable properties.