3.5 Intersection theory

The previous corollary allows us to make the following definition:

Definition 3.31. Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps with X and Z compact, and $\dim X + \dim Z = \dim Y$. Then we define the (mod 2) intersection number of f and g to be

$$I_2(f, g) = \#(X_{f'} \times_g Z) \pmod 2,$$

where $f' : X \longrightarrow Y$ is any smooth map smoothly homotopic to f but transverse to g.

Example 3.32. If C_1, C_2 are two distinct great circles on S^2 then they have two transverse intersection points, so $I_2(C_1, C_2) = 0$ in $\mathbb{Z}_2$. Of course we can shrink one of the circles to get a homotopic one which does not intersect the other at all. This corresponds to the standard cobordism from two points to the empty set.

Example 3.33. If (e_1, e_2, e_3) is a basis for $\mathbb{R}^3$ we can consider the following two embeddings of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ into $\mathbb{R}P^2$: $\iota_1 : \theta \mapsto \langle \cos(\theta/2)e_1 + \sin(\theta/2)e_2 \rangle$ and $\iota_2 : \theta \mapsto \langle \cos(\theta/2)e_2 + \sin(\theta/2)e_3 \rangle$. These two embedded submanifolds intersect transversally in a single point $\langle e_2 \rangle$, and hence $I_2(\iota_1, \iota_2) = 1$ in $\mathbb{Z}_2$. As a result, there is no way to deform ι_i so that they intersect transversally in zero points.

Example 3.34. Given a smooth map $f : X \longrightarrow Y$ for X compact and $\dim Y = 2 \dim X$, we may consider the self-intersection $I_2(f, f)$. In the previous examples we may check $I_2(C_1, C_1) = 0$ and $I_2(\iota_1, \iota_1) = 1$. Any embedded S^1 in an oriented surface has no self-intersection. If the surface is nonorientable, the self-intersection may be nonzero.

Example 3.35. Let $p \in S^1$. Then the identity map $\mathrm{Id} : S^1 \longrightarrow S^1$ is transverse to the inclusion $\iota : p \longrightarrow S^1$ with one point of intersection. Hence the identity map is not (smoothly) homotopic to a constant map, which would be transverse to ι with zero intersection. Using smooth approximation, get that Id is not continuously homotopic to a constant map, and also that S^1 is not contractible.

Example 3.36. By the previous argument, any compact manifold is not contractible.

Example 3.37. Consider $SO(3) \cong \mathbb{R}P^3$ and let $\ell \subset \mathbb{R}P^3$ be a line, diffeomorphic to S^1. This line corresponds to a path of rotations about an axis by $\theta \in [0, \pi]$ radians. Let $\mathcal{P} \subset \mathbb{R}P^3$ be a plane intersecting ℓ in one point. Since this is a transverse intersection in a single point, ℓ cannot be deformed to a point (which would have zero intersection with $\mathcal{P}$. This shows that the path of rotations is not homotopic to a constant path.

If $\iota : \theta \mapsto \iota(\theta)$ is the embedding of S^1, then traversing the path twice via $\iota' : \theta \mapsto \iota(2\theta)$, we obtain a map ι' which is transverse to $\mathcal{P}$ but with two intersection points. Hence it is possible that ι' may be deformed so as not to intersect $\mathcal{P}$. Can it be done?

Example 3.38. Consider $\mathbb{R}P^4$ and two transverse hyperplanes P_1, P_2 each an embedded copy of $\mathbb{R}P^3$. These then intersect in $P_1 \cap P_2 = \mathbb{R}P^2$, and since $\mathbb{R}P^2$ is not null-homotopic, we cannot deform the planes to remove all intersection.

Intersection theory also allows us to define the degree of a map modulo 2. The degree measures how many generic preimages there are of a local diffeomorphism.

Definition 3.39. Let $f : M \longrightarrow N$ be a smooth map of manifolds of the same dimension, and suppose M is compact and N connected. Let $p \in N$ be any point. Then we define $\deg_2(f) = I_2(f, p)$.

Example 3.40. Let $f : S^1 \longrightarrow S^1$ be given by $z \mapsto z^k$. Then $\deg_2(f) = k$ (mod 2).

Example 3.41. If $p : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$ is a polynomial of degree k, then as a map $S^2 \longrightarrow S^2$ we have $\deg_2(p) = k$ (mod 2), and hence any odd polynomial has at least one root. To get the fundamental theorem of algebra, we must consider *oriented cobordism*

Even if submanifolds C, C' do not intersect, it may be that there are more sophisticated geometrical invariants which cause them to be "intertwined" in some way. One example of this is linking number.

Definition 3.42. Suppose that $M, N \subset \mathbb{R}^{k+1}$ are compact embedded submanifolds with $\dim M + \dim N = k$, and let us assume they are transverse, meaning they do not intersect at all.

Then define $\lambda : M \times N \longrightarrow S^k$ via

$$(x, y) \mapsto \frac{x - y}{|x - y|}.$$

Then we define the (mod 2) linking number of M, N to be $\deg_2(\lambda)$.

Example 3.43. Consider the standard Hopf link in $\mathbb{R}^3$. Then it is easy to calculate that $\deg_2(\lambda) = 1$. On the other hand, the standard embedding of disjoint circles (differing by a translation, say) has $\deg_2(\lambda) = 0$. Hence it is impossible to deform the circles through embeddings of $S^1 \sqcup S^1 \longrightarrow \mathbb{R}^3$, so that they are unlinked. Why must we stay within the space of embeddings, and not allow the circles to intersect?

3.6 Partitions of unity and Whitney embedding

Partitions of unity allow us to *go from local to global*, i.e. to build a global object on a manifold by building it on each open set of a cover, smoothly tapering each local piece so it is compactly supported in each open set, and then taking a sum over open sets. This is a very flexible operation which uses the properties of smooth functions—it will not work for complex manifolds, for example. Our main example of such a passage from local to global is to build a global map from a manifold to $\mathbb{R}^N$ which is an embedding, a result first proved by Whitney.

Definition 3.44. A collection of subsets $\{U_\alpha\}$ of the topological space M is called *locally finite* when each point $x \in M$ has a neighbourhood V intersecting only finitely many of the U_α.

Definition 3.45. A covering $\{V_\alpha\}$ is a *refinement* of the covering $\{U_\beta\}$ when each V_α is contained in some U_β.

Lemma 3.46. *Any open covering $\{A_\alpha\}$ of a topological manifold has a countable, locally finite refinement $\{(U_i, \varphi_i)\}$ by coordinate charts such that $\varphi_i(U_i) = B(0,3)$ and $\{V_i = \varphi_i^{-1}(B(0,1))\}$ is still a covering of M. We will call such a cover a* regular *covering. In particular, any topological manifold is* paracompact *(i.e. every open cover has a locally finite refinement)*

Proof. If M is compact, the proof is easy: choosing coordinates around any point $x \in M$, we can translate and rescale to find a covering of M by a refinement of the type desired, and choose a finite subcover, which is obviously locally finite.

For a general manifold, we note that by second countability of M, there is a countable basis of coordinate neighbourhoods and each of these charts is a countable union of open sets P_i with $\overline{P_i}$ compact. Hence M has a countable basis $\{P_i\}$ such that $\overline{P_i}$ is compact.

Using these, we may define an increasing sequence of compact sets which exhausts M: let $K_1 = \overline{P_1}$, and

$$K_{i+1} = \overline{P_1 \cup \cdots \cup P_r},$$

where $r > 1$ is the first integer with $K_i \subset P_1 \cup \cdots \cup P_r$.

Now note that M is the union of ring-shaped sets $K_i \backslash K_{i-1}^\circ$, each of which is compact. If $p \in A_\alpha$, then $p \in K_{i+1} \backslash K_i^\circ$ for some i. Now choose a coordinate neighbourhood $(U_{p,\alpha}, \varphi_{p,\alpha})$ with $U_{p,\alpha} \subset K_{i+2} \backslash K_{i-1}^\circ$ and $\varphi_{p,\alpha}(U_{p,\alpha}) = B(0,3)$ and define $V_{p,\alpha} = \varphi^{-1}(B(0,1))$.

Letting p, α vary, these neighbourhoods cover the compact set $K_{i+1} \backslash K_i^\circ$ without leaving the band $K_{i+2} \backslash K_{i-1}^\circ$. Choose a finite subcover $V_{i,k}$ for each i. Then $(U_{i,k}, \varphi_{i,k})$ is the desired locally finite refinement. □

Definition 3.47. A smooth partition of unity is a collection of smooth non-negative functions $\{f_\alpha : M \longrightarrow \mathbb{R}\}$ such that

i) $\{\mathrm{supp} f_\alpha = \overline{f_\alpha^{-1}(\mathbb{R}\backslash\{0\})}\}$ is locally finite,

ii) $\sum_\alpha f_\alpha(x) = 1 \;\; \forall x \in M$, hence the name.

A partition of unity is *subordinate* to an open cover $\{U_i\}$ when $\forall \alpha$, $\mathrm{supp} f_\alpha \subset U_i$ for some i.

Theorem 3.48. *Given a regular covering $\{(U_i, \varphi_i)\}$ of a manifold, there exists a partition of unity $\{f_i\}$ subordinate to it with $f_i > 0$ on V_i and $\mathrm{supp} f_i \subset \varphi_i^{-1}(\overline{B(0,2)})$.*

Proof. A *bump function* is a smooth non-negative real-valued function $\tilde{g}$ on $\mathbb{R}^n$ with $\tilde{g}(x) = 1$ for $||x|| \leq 1$ and $\tilde{g}(x) = 0$ for $||x|| \geq 2$. For instance, take

$$\tilde{g}(x) = \frac{h(2 - ||x||)}{h(2 - ||x||) + h(||x|| + 1)},$$

for $h(t)$ given by $e^{-1/t}$ for $t > 0$ and 0 for $t < 0$.

Having this bump function, we can produce non-negative bump functions on the manifold $g_i = \tilde{g} \circ \varphi_i$ which have support $\mathrm{supp} g_i \subset \varphi_i^{-1}(\overline{B(0,2)})$ and take the value $+1$ on $\overline{V_i}$. Finally we define our partition of unity via

$$f_i = \frac{g_i}{\sum_j g_j}, \quad i = 1, 2, \ldots.$$

$\square$

We now investigate the embedding of arbitrary smooth manifolds as regular submanifolds of $\mathbb{R}^k$. We shall first show by a straightforward argument that any smooth manifold may be embedded in some $\mathbb{R}^N$ for some sufficiently large N. We will then explain how to cut down on N and approach the optimal $N = 2 \dim M$ which Whitney showed (we shall reach $2 \dim M + 1$ and possibly at the end of the course, show $N = 2 \dim M$.)

Theorem 3.49 (Compact Whitney embedding in $\mathbb{R}^N$). *Any compact manifold may be embedded in $\mathbb{R}^N$ for sufficiently large N.*

Proof. Let $\{(U_i \supset V_i, \varphi_i)\}_{i=1}^k$ be a *finite* regular covering, which exists by compactness. Choose a partition of unity $\{f_1, \ldots, f_k\}$ as in Theorem 3.48 and define the following "zoom-in" maps $M \longrightarrow \mathbb{R}^{\dim M}$:

$$\tilde{\varphi}_i(x) = \begin{cases} f_i(x)\varphi_i(x) & x \in U_i, \\ 0 & x \notin U_i. \end{cases}$$

Then define a map $\Phi : M \longrightarrow \mathbb{R}^{k(\dim M + 1)}$ which zooms simultaneously into all neighbourhoods, with extra information to guarantee injectivity:

$$\Phi(x) = (\tilde{\varphi}_1(x), \ldots, \tilde{\varphi}_k(x), f_1(x), \ldots, f_k(x)).$$

Note that $\Phi(x) = \Phi(x')$ implies that for some i, $f_i(x) = f_i(x') \neq 0$ and hence $x, x' \in U_i$. This then implies that $\varphi_i(x) = \varphi_i(x')$, implying $x = x'$. Hence Φ is injective.

We now check that $D\Phi$ is injective, which will show that it is an injective immersion. At any point x the differential sends $v \in T_x M$ to the following vector in $\mathbb{R}^{\dim M} \times \cdots \times \mathbb{R}^{\dim M} \times \mathbb{R} \times \cdots \times \mathbb{R}$.

$$(Df_1(v)\varphi_1(x) + f_1(x)D\varphi_1(v), \ldots, Df_k(v)\varphi_k(x) + f_k(x)D\varphi_1(v), Df_1(v), \ldots, Df_k(v)$$

But this vector cannot be zero. Hence we see that Φ is an immersion.

But an injective immersion from a compact space must be an embedding: view Φ as a bijection onto its image. We must show that Φ^{-1} is continuous, i.e. that Φ takes closed sets to closed sets. If $K \subset M$ is closed, it is also compact and hence $\Phi(K)$ must be compact, hence closed (since the target is Hausdorff). $\qquad\square$

Theorem 3.50 (Compact Whitney embedding in $\mathbb{R}^{2n+1}$). *Any compact n-manifold may be embedded in $\mathbb{R}^{2n+1}$.*

Proof. Begin with an embedding $\Phi : M \longrightarrow \mathbb{R}^N$ and assume $N > 2n + 1$. We then show that by projecting onto a hyperplane it is possible to obtain an embedding to $\mathbb{R}^{N-1}$.

A vector $v \in S^{N-1} \subset \mathbb{R}^N$ defines a hyperplane (the orthogonal complement) and let $P_v : \mathbb{R}^N \longrightarrow \mathbb{R}^{N-1}$ be the orthogonal projection to this hyperplane. We show that the set of v for which $\Phi_v = P_v \circ \Phi$ fails to be an embedding is a set of measure zero, hence that it is possible to choose v for which Φ_v is an embedding.

Φ_v fails to be an embedding exactly when Φ_v is not injective or $D\Phi_v$ is not injective at some point. Let us consider the two failures separately:

If v is in the image of the map $\beta_1 : (M \times M) \backslash \Delta_M \longrightarrow S^{N-1}$ given by

$$\beta_1(p_1, p_2) = \frac{\Phi(p_2) - \Phi(p_1)}{||\Phi(p_2) - \Phi(p_1)||},$$

then Φ_v will fail to be injective. Note however that β_1 maps a $2n$-dimensional manifold to a $N-1$-manifold, and if $N > 2n + 1$ then baby Sard's theorem implies the image has measure zero.

The immersion condition is a local one, which we may analyze in a chart (U, φ). Φ_v will fail to be an immersion in U precisely when v coincides with a vector in the normalized image of $D(\Phi \circ \varphi^{-1})$ where

$$\Phi \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N.$$

Hence we have a map (letting $N(w) = ||w||$)

$$\frac{D(\Phi \circ \varphi^{-1})}{N \circ D(\Phi \circ \varphi^{-1})} : U \times S^{n-1} \longrightarrow S^{N-1}.$$

The image has measure zero as long as $2n - 1 < N - 1$, which is certainly true since $2n < N - 1$. Taking union over countably many charts, we see that immersion fails on a set of measure zero in S^{N-1}.

Hence we see that Φ_v fails to be an embedding for a set of $v \in S^{N-1}$ of measure zero. Hence we may reduce N all the way to $N = 2n + 1$. $\qquad\square$

Corollary 3.51. *We see from the proof that if we do not require injectivity but only that the manifold be* immersed *in $\mathbb{R}^N$, then we can take $N = 2n$ instead of $2n + 1$.*

Theorem 3.52 (noncompact Whitney embedding in $\mathbb{R}^{2n+1}$). *Any smooth n-manifold may be embedded in $\mathbb{R}^{2n+1}$ (or immersed in $\mathbb{R}^{2n}$).*

Proof. We saw that any manifold may be written as a countable union of increasing compact sets $M = \cup K_i$, and that a regular covering $\{(U_{i,k} \supset V_{i,k}, \varphi_{i,k})\}$ of M can be chosen so that for fixed i, $\{V_{i,k}\}_k$ is a finite cover of $K_{i+1}\backslash K_i^\circ$ and each $U_{i,k}$ is contained in $K_{i+2}\backslash K_{i-1}^\circ$.

This means that we can express M as the union of 3 open sets W_0, W_1, W_2, where
$$W_j = \bigcup_{i \equiv j(\mathrm{mod}3)} (\cup_k U_{i,k}).$$

Each of the sets $R_i = \cup_k U_{i,k}$ may be injectively immersed in $\mathbb{R}^{2n+1}$ by the argument for compact manifolds, since they have a finite regular cover. Call these injective immersions $\Phi_i : R_i \longrightarrow \mathbb{R}^{2n+1}$. The image $\Phi_i(R_i)$ is bounded since all the charts are, by some radius r_i. The open sets R_i, $i \equiv j(\mathrm{mod}3)$ for fixed j are disjoint, and by translating each Φ_i, $i \equiv j(\mathrm{mod}3)$ by an appropriate constant, we can ensure that their images in $\mathbb{R}^{2n+1}$ are disjoint as well.

Let $\Phi_i' = \Phi_i + (2(r_{i-1} + r_{i-2} + \cdots) + r_i)\overrightarrow{e}_1$. Then $\Psi_j = \cup_{i \equiv j(\mathrm{mod}3)} \Phi_i' : W_j \longrightarrow \mathbb{R}^{2n+1}$ is an embedding.

Now that we have injective immersions Ψ_0, Ψ_1, Ψ_2 of W_0, W_1, W_2 in $\mathbb{R}^{2n+1}$, we may use the original argument for compact manifolds: Take the partition of unity subordinate to $U_{i,k}$ and resum it, obtaining a 3-element partition of unity $\{f_1, f_2, f_3\}$, with $f_j = \sum_{i \equiv j(\mathrm{mod}3)} \sum_k f_{i,k}$. Then the map
$$\Psi = (f_1\Psi_1, f_2\Psi_2, f_3\Psi_3, f_1, f_2, f_3)$$
is an injective immersion of M into $\mathbb{R}^{6n+3}$. To see that it is in fact an embedding, note that any closed set $C \subset M$ may be written as a union of closed sets $C = C_1 \cup C_2 \cup C_3$, where $C_j = \cup_{i \equiv j(\mathrm{mod}3)}(C \cap K_{i+1}\backslash K_i^\circ)$ is a disjoint union of compact sets. Ψ is injective, hence C_j is mapped to a disjoint union of compact sets, hence a closed set. Then $\Psi(C)$ is a union of 3 closed sets, hence closed, as required.

Using projection to hyperplanes we may again reduce to $\mathbb{R}^{2n+1}$, but if we exclude all hyperplanes perpendicular to $\mathrm{Span}((e_1, 0, 0, 0, 0, 0), (0, e_1, 0, 0, 0, 0), (0, 0, e_1, 0, 0, 0))$, we obtain an injective immersion Ψ' which is *proper*, meaning that inverse images of compact sets are compact. This space of forbidden planes has measure zero as long as $N - 1 > 3$, so that we may reduce to $2n + 1$ for $n > 1$. We leave as an exercise the $n = 1$ case (or see Bredon for a slightly different proof).

The fact that the resulting injective immersion Ψ' is proper implies that it is an embedding, by the closed map lemma, as follows. $\qquad \square$

Lemma 3.53 (Closed map lemma for proper maps)**.** *Let $f : X \longrightarrow Y$ be a proper continuous map of topological manifolds. Then f is a closed map.*

Proof. Let $K \subset X$ be closed; we show that $f(K)$ contains all its limit points and hence is closed. Let $y \in Y$ be a limit point for $f(K)$. Choose a precompact neighbourhood U of y, so that y is also a limit point of $f(K) \cap \overline{U}$. Since f is proper, $f^{-1}(\overline{U})$ is compact, and hence $K \cap f^{-1}(\overline{U})$ is compact as well. But then by continuity, $f(K \cap f^{-1}(\overline{U})) = f(K) \cap \overline{U}$ is compact, implying it is closed. Hence $y \in f(K) \cap \overline{U} \subset f(K)$, as required. $\qquad \square$

We now use Whitney embedding to prove the existence of tubular neighbourhoods for submanifolds of $\mathbb{R}^N$, a key point in proving genericity of transversality. Tubular neighbourhoods also exist for submanifolds of any manifold, but we leave this corollary for the reader.

If $Y \subset \mathbb{R}^N$ is an embedded submanifold, the normal space at $y \in Y$ is defined by $N_y Y = \{v \in \mathbb{R}^N \ : \ v \perp T_y Y\}$. The collection of all normal spaces of all points in Y is called the normal bundle:

$$NY = \{(y, v) \in Y \times \mathbb{R}^N \ : \ v \in N_y Y\}.$$

Proposition 3.54. $NY \subset \mathbb{R}^N \times \mathbb{R}^N$ *is an embedded submanifold of dimension N.*

Proof. Given $y \in Y$, choose coordinates $(u^1, \ldots u^N)$ in a neighbourhood $U \subset \mathbb{R}^N$ of y so that $Y \cap U = \{u^{n+1} = \cdots = u^N = 0\}$. Define $\Phi : U \times \mathbb{R}^N \longrightarrow \mathbb{R}^{N-n} \times \mathbb{R}^n$ via

$$\Phi(x, v) = (u^{n+1}(x), \ldots, u^N(x), \langle v, \tfrac{\partial}{\partial u^1}|_x \rangle, \ldots, \langle v, \tfrac{\partial}{\partial u^n}|_x \rangle),$$

so that $\Phi^{-1}(0)$ is precisely $NY \cap (U \times \mathbb{R}^N)$. We then show that 0 is a regular value: observe that, writing v in terms of its components $v^j \frac{\partial}{\partial x^j}$ in the standard basis for $\mathbb{R}^N$,

$$\langle v, \tfrac{\partial}{\partial u^i}|_x \rangle = \langle v^j \tfrac{\partial}{\partial x^j}, \tfrac{\partial x^k}{\partial u^i}(u(x)) \tfrac{\partial}{\partial x^k}|_x \rangle = \sum_{j=1}^N v^j \tfrac{\partial x^j}{\partial u^i}(u(x))$$

Therefore the Jacobian of Φ is the $((N-n) + n) \times (N + N)$ matrix

$$D\Phi(x) = \begin{pmatrix} \frac{\partial u^j}{\partial x^i}(x) & 0 \\ * & \frac{\partial x^j}{\partial u^i}(u(x)) \end{pmatrix}$$

The N rows of this matrix are linearly independent, proving Φ is a submersion. $\square$

The normal bundle NY contains $Y \cong Y \times \{0\}$ as a regular submanifold, and is equipped with a smooth map $\pi : NY \longrightarrow Y$ sending $(y, v) \mapsto y$. The map π is a surjective submersion and is the bundle projection. The vector spaces $\pi^{-1}(y)$ for $y \in Y$ are called the fibers of the bundle and NY is an example of a vector bundle.

We may take advantage of the embedding in $\mathbb{R}^N$ to define a smooth map $E : NY \longrightarrow \mathbb{R}^N$ via

$$E(x, v) = x + v.$$

Definition 3.55. A tubular neighbourhood of the embedded submanifold $Y \subset \mathbb{R}^N$ is a neighbourhood U of Y in $\mathbb{R}^N$ that is the diffeomorphic image under E of an open subset $V \subset NY$ of the form

$$V = \{(y, v) \in NY \ : \ |v| < \delta(y)\},$$

for some positive continuous function $\delta : M \longrightarrow \mathbb{R}$.

If $U \subset \mathbb{R}^N$ is such a tubular neighbourhood of Y, then there does exist a positive continuous function $\epsilon : Y \longrightarrow \mathbb{R}$ such that $U_\epsilon = \{x \in \mathbb{R}^N : \exists y \in Y \text{ with } |x - y| < \epsilon(y)\}$ is contained in U. This is simply

$$\epsilon(y) = \sup\{r : B(y, r) \subset U\},$$

which is continuous since $\forall \epsilon > 0, \exists x \in U$ for which $\epsilon(y) \leq |x - y| + \epsilon$. For any other $y' \in Y$, this is $\leq |y - y'| + |x - y'| + \epsilon$. Since $|x - y'| \leq \epsilon(y')$, we have $|\epsilon(y) - \epsilon(y')| \leq |y - y'| + \epsilon$.

Theorem 3.56 (Tubular neighbourhood theorem). *Every regular submanifold of $\mathbb{R}^N$ has a tubular neighbourhood.*

Proof. First we show that E is a local diffeomorphism near $y \in Y \subset NY$. if ι is the embedding of Y in $\mathbb{R}^N$, and $\iota' : Y \longrightarrow NY$ is the embedding in the normal bundle, then $E \circ \iota' = \iota$, hence we have $DE \circ D\iota' = D\iota$, showing that the image of $DE(y)$ contains $T_y Y$. Now if ι is the embedding of $N_y Y$ in $\mathbb{R}^N$, and $\iota' : N_y Y \longrightarrow NY$ is the embedding in the normal bundle, then $E \circ \iota' = \iota$. Hence we see that the image of $DE(y)$ contains $N_y Y$, and hence the image is all of $T_y \mathbb{R}^N$. Hence E is a diffeomorphism on some neighbourhood

$$V_\delta(y) = \{(y', v') \in NY : |y' - y| < \delta, \ |v'| < \delta\}, \quad \delta > 0.$$

Now for $y \in Y$ let $r(y) = \sup\{\delta : E|_{V_\delta(y)} \text{ is a diffeomorphism}\}$ if this is ≤ 1 and let $r(y) = 1$ otherwise. The function $r(y)$ is continuous, since if $|y - y'| < r(y)$, then $V_\delta(y') \subset V_{r(y)}(y)$ for $\delta = r(y) - |y - y'|$. This means that $r(y') \geq \delta$, i.e. $r(y) - r(y') \leq |y - y'|$. Switching y and y', this remains true, hence $|r(y) - r(y')| \leq |y - y'|$, yielding continuity.

Finally, let $V = \{(y, v) \in NY : |v| < \frac{1}{2} r(y)\}$. We show that E is injective on V. Suppose $(y, v), (y', v') \in V$ are such that $E(y, v) = E(y', v')$, and suppose wlog $r(y') \leq r(y)$. Then since $y + v = y' + v'$, we have

$$|y - y'| = |v - v'| \leq |v| + |v'| \leq \tfrac{1}{2} r(y) + \tfrac{1}{2} r(y') \leq r(y).$$

Hence y, y' are in $V_{r(y)}(y)$, on which E is a diffeomorphism. The required tubular neighbourhood is then $U = E(V)$. $\square$