4 Vector fields

4.1 Derivations

The space $C^\infty(M,\mathbb{R})$ of smooth functions on M is not only a vector space but also a ring, with multiplication $(fg)(p) := f(p)g(p)$. That this defines a smooth function is clear from the fact that it is a composition of the form

$$M \xrightarrow{\Delta} M \times M \xrightarrow{f\times g} \mathbb{R}\times\mathbb{R} \xrightarrow{m} \mathbb{R} \, .$$

Given a smooth map $\varphi : M \longrightarrow N$ of manifolds, we obtain a natural operation $\varphi^* : C^\infty(N,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$, given by $f \mapsto f\circ\varphi$. This is called the pullback of functions, and defines a homomorphism of rings since $\Delta\circ\varphi = (\varphi\times\varphi)\circ\Delta$.

The association $M \mapsto C^\infty(M,\mathbb{R})$ and $\varphi \mapsto \varphi^*$ is therefore a *contravariant* functor from the category of manifolds to the category of rings, and is the basis for algebraic geometry, the algebraic representation of geometrical objects.

It is easy to see from this that any diffeomorphism $\varphi : M \longrightarrow M$ defines an automorphism φ^* of $C^\infty(M,\mathbb{R})$, but actually all automorphisms are of this form (Exercise!).

The concept of derivation of an algebra A is the infinitesimal version of an automorphism of A. That is, if $\phi_t : A \longrightarrow A$ is a family of automorphisms of A starting at Id, so that $\phi_t(ab) = \phi_t(a)\phi_t(b)$, then the map $a \mapsto \frac{d}{dt}|_{t=0}\phi_t(a)$ is a derivation.

Definition 4.1. A derivation of the $\mathbb{R}$-algebra A is a $\mathbb{R}$-linear map $D : A \longrightarrow A$ such that $D(ab) = (Da)b + a(Db)$. The space of all derivations is denoted $\mathrm{Der}(A)$.

If automorphisms of $C^\infty(M,\mathbb{R})$ correspond to diffeomorphisms, then it is natural to ask what derivations correspond to. We now show that they correspond to vector fields.

The vector fields $\Gamma^\infty(M,TM)$ form a vector space over $\mathbb{R}$ of infinite dimension (unless M is a finite set). They also form a module over the ring of smooth functions $C^\infty(M,\mathbb{R})$ via pointwise multiplication: for $f \in C^\infty(M,\mathbb{R})$ and $X \in \Gamma^\infty(M,TM)$, $fX : x \mapsto f(x)X(x)$ is a smooth vector field (why?)

The important property of vector fields which we are interested in is that they act as $\mathbb{R}$-derivations of the algebra of smooth functions. Locally, it is clear that a vector field $X = \sum_i a^i \frac{\partial}{\partial x^i}$ gives a derivation of the algebra of smooth functions, via the formula $X(f) = \sum_i a^i \frac{\partial f}{\partial x^i}$, since

$$X(fg) = \sum_i a^i\left(\frac{\partial f}{\partial x^i} g + f\frac{\partial g}{\partial x^i}\right) = X(f)g + fX(g).$$

We wish to verify that this local action extends to a well-defined global derivation on $C^\infty(M,\mathbb{R})$.

Definition 4.2. The differential of a function $f \in C^\infty(M,\mathbb{R})$ is the function on TM given by composing $Tf : TM \to T\mathbb{R}$ with the second projection $p_2 : T\mathbb{R} = \mathbb{R}\times\mathbb{R} \to \mathbb{R}$:

$$df = p_2 \circ Tf \tag{79}$$

Recall that if (U, φ) is a chart for M, then $(TU, D\varphi)$ is a chart for TM. More explicitly, if $(x_1, \ldots, x_n)$ is the coordinate system on U given by φ, then the induced coordinate system on TU is $(x_1 \circ \pi, \ldots, x_n \circ \pi, dx_1, \ldots, dx_n)$. Often, we omit the bundle projection π and we write ξ_i for the differential dx_i, and so the induced coordinates are $(x_1, \ldots, x_n, \xi_1, \ldots, x_n)$.

Definition 4.3. Let $X \in \Gamma(M, TM)$ be a vector field. Then we define

$$X(f) = df \circ X.$$

This is called the directional (or Lie) derivative of f along X.

In coordinates, if $X = \sum a_i \partial/\partial x_i$, then $X(f) = \sum a_i \partial f/\partial x_i$, coinciding with the usual directional derivative mentioned above. This shows that $f \mapsto X(f)$ has the derivation property (since it satisfies it locally), but we can alternatively see that it is a derivation by using the property

$$d(fg) = fdg + gdf$$

of the differential of a product (here fdg is really $\pi^* fdg$).

Theorem 4.4. *The map $X \mapsto (f \mapsto X(f))$ is an isomorphism*

$$\Gamma(M, TM) \to \mathit{Der}(C^\infty(M, \mathbb{R})).$$

Proof. First we prove the result for an open set $U \subset \mathbb{R}^n$. Let D be a derivation of $C^\infty(U, \mathbb{R})$ and define the smooth functions $a^i = D(x^i)$. Then we claim $D = \sum_i a^i \frac{\partial}{\partial x^i}$. We prove this by testing against smooth functions. Any smooth function f on $\mathbb{R}^n$ may be written

$$f(x) = f(0) + \sum_i x^i g_i(x),$$

with $g_i(0) = \frac{\partial f}{\partial x^i}(0)$ (simply take $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx)dt$). Translating the origin to $y \in U$, we obtain for any $z \in U$

$$f(z) = f(y) + \sum_i (x^i(z) - x^i(y))g_i(z), \qquad g_i(y) = \tfrac{\partial f}{\partial x^i}(y).$$

Applying D, we obtain

$$Df(z) = \sum_i (Dx^i)g_i(z) - \sum_i (x^i(z) - x^i(y))Dg_i(z).$$

Letting z approach y, we obtain

$$Df(y) = \sum_i a^i \tfrac{\partial f}{\partial x^i}(y) = X(f)(y),$$

as required.

To prove the global result, let $(V_i \subset U_i, \varphi_i)$ be a regular covering and θ_i an associated partition of unity. Then for each i, $\theta_i D : f \mapsto \theta_i D(f)$ is also a derivation of $C^\infty(M, \mathbb{R})$. This derivation defines a unique derivation D_i of $C^\infty(U_i, \mathbb{R})$ such that $D_i(f|_{U_i}) = (\theta_i Df)|_{U_i}$, since for any point $p \in U_i$, a given function $g \in C^\infty(U_i, \mathbb{R})$ may be replaced with a function

$\tilde{g} \in C^\infty(M, \mathbb{R})$ which agrees with g on a small neighbourhood of p, and we define $(D_i g)(p) = \theta_i(p) D\tilde{g}(p)$. This definition is independent of $\tilde{g}$, since if $h_1 = h_2$ on an open set W, $Dh_1 = Dh_2$ on that open set (let $\psi = 1$ in a neighbourhood of p and vanish outside W; then $h_1 - h_2 = (h_1 - h_2)(1 - \psi)$ and applying D we obtain zero in W).

The derivation D_i is then represented by a vector field X_i, which must vanish outside the support of θ_i. Hence it may be extended by zero to a global vector field which we also call X_i. Finally we observe that for $X = \sum_i X_i$, we have

$$X(f) = \sum_i X_i(f) = \sum_i D_i(f) = D(f),$$

□

4.2 Flows

Since vector fields are derivations, we have a natural source of examples, coming from infinitesimal automorphisms of M:

Example 4.5. Let φ_t : be a smooth family of maps $M \to M$ with $\varphi_0 = \mathrm{Id}$. That is, let $\varphi : (-\epsilon, \epsilon) \times M \longrightarrow M$ be smooth with $\varphi \circ j_0 = \mathrm{id}$, for $j_t(x) = (t, x)$. Then $X(f)(p) = \frac{d}{dt}|_{t=0}(\varphi_t^* f)(p)$ defines a smooth vector field. A better way of seeing it is to rewrite it as follows: Let $\frac{\partial}{\partial t}$ be the coordinate vector field on $(-\epsilon, \epsilon)$ and observe

$$X(f) = \tfrac{\partial}{\partial t}(\varphi^* f) \circ j_0.$$

Remark 4.6. A better way of describing the vector field from Example 4.5 is to note that the pullback of $T\varphi$ by j_0 is a bundle map from $j_0^* TU$ to TM over the identity map $M \to M$, and then X is simply the image of $j_0^* \frac{\partial}{\partial t}$ under $j_0^* T\varphi$, or informally

$$X = \varphi_*|_{t=0}(\tfrac{\partial}{\partial t}) \tag{80}$$

Essentially, a smooth vector field may always be expressed in this way, i.e. as the derivative of a family of automorphisms of M. The only caveat is that ϵ must be allowed to vary along the manifold M if it is noncompact. This gives rise to the notion of a "local 1-parameter group of diffeomorphisms", as follows:

Definition 4.7. A local 1-parameter group of diffeomorphisms is an open set $U \subset \mathbb{R} \times M$ containing $\{0\} \times M$ and a smooth map

$$\begin{aligned} \Phi : &U \longrightarrow M \\ &(t, x) \mapsto \varphi_t(x) \end{aligned}$$

such that $\mathbb{R} \times \{x\} \cap U$ is connected, $\varphi_0(x) = x$ for all x and if $(t, x), (t + t', x), (t', \varphi_t(x))$ are all in U then $\varphi_{t'}(\varphi_t(x)) = \varphi_{t+t'}(x)$.

The derivative (80) of this family of diffeomorphisms is a vector field X, and we say that Φ is the *flow* of X.

Then the local existence and uniqueness of solutions to systems of ODE implies that every smooth vector field $X \in \Gamma(M, TM)$ gives rise to a local 1-parameter group of diffeomorphisms (U, Φ) such that the curve $\gamma_x : t \mapsto \varphi_t(x)$ satisfies $(\gamma_x)_*(\frac{d}{dt}) = X(\gamma_x(t))$ (this means that γ_x is an integral curve or "trajectory" of the "dynamical system" defined by X). Furthermore, if (U', Φ') are another such data, then $\Phi = \Phi'$ on $U \cap U'$.

Remark 4.8. We can rephrase the system of ODEs as the initial value problem

$$\Phi_* \tfrac{\partial}{\partial t} = X,$$
$$\Phi \circ j_0 = \mathrm{id}_M.$$

This makes it very clear that 80 holds. In fact, the existence and uniqueness theorem is slightly more general, in that it allows the vector field to depend on time, so that X may be defined on $\mathbb{R} \times X$ with vanishing first projection $Tp_1(X) = 0$, and Φ may then be extended to a map $\tilde{\Phi} : U \to \mathbb{R} \times M$ with the property $\tilde{\Phi}_* \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + X$ and $\tilde{\Phi} \circ j_0 = j_0$. Uniqueness is then the statement that if $X = 0$ then $\tilde{\Phi}$ must be the identity.

Definition 4.9. A vector field $X \in \Gamma(M, TM)$ is called *complete* when it generates a global 1-parameter group of diffeomorphisms, i.e. $U = \mathbb{R} \times M$ in the above discussion.

We omit the proof of the following theorem, though it is not difficult to show that if $[0, \omega)$ is the maximal interval on which a trajectory γ is defined for non-negative times, and if the image $\gamma([0, \omega))$ has compact closure in M, then ω must be infinity.

Theorem 4.10. *If M is compact, then every smooth vector field is complete.*

Example 4.11. The vector field $X = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$ is not complete. For initial condition x_0, have integral curve $\gamma(t) = x_0(1 - tx_0)^{-1}$, which gives $\Phi(t, x_0) = x_0(1 - tx_0)^{-1}$, which is well-defined on $\{1 - tx > 0\}$.

4.3 Commuting flows

Given two derivations D_1, D_2 of an algebra, the commutator $[D_1, D_2]$ is another derivation. In fact, if D_1 and D_2 arise from families of automorphisms φ_t, ψ_t respectively (with $\varphi_0 = \psi_0 = \mathrm{id}$), then the family of automorphisms $\varphi_t \psi_t \varphi_t^{-1} \psi_t^{-1}$ has zero first derivative but has second derivative given by $[D_1, D_2]$. This explains why derivations, or infinitesimal symmetries, always have the structure of a Lie algebra.

Using the correspondence between $\Gamma(M, TM)$ and $\mathrm{Der}(C^\infty(M, \mathbb{R}))$, we see that vector fields are endowed with a Lie bracket, given simply by their commutator when viewed as derivations.

Example 4.12. Let $X = \sum \alpha_i \partial_i$ and $Y = \sum \beta_i \partial_i$ be vector fields in coordinates. Then the Lie bracket $[X, Y] = \sum \gamma_i \partial_i$, where

$$\begin{aligned} \gamma_i &= X(Y(x_i)) - Y(X(x_i)) \\ &= X(\beta_i) - Y(\alpha_i) \\ &= \sum (\alpha_k \partial_k \beta_i - \beta_k \partial_k \alpha_i). \end{aligned} \tag{81}$$

The usefulness of the Lie bracket is clear from the fact that if X, Y are vector fields generating flows φ_t, ψ_s respectively, then it follows that $[X, Y]$ coincides with the time derivative of the family of vector fields $(T\varphi_{-t}) \circ Y \circ \varphi_t)$ at $t = 0$, and if $[X, Y] = 0$, then this guarantees $(\varphi_t)_* Y = Y$, and therefore that φ_t commutes with ψ_s.

Lemma 4.13. *If Φ is the flow of X, then $\Phi_* X = X$, i.e. $(\varphi_t)_* X = X$ over the appropriate domain.*

Proof. Let $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the sum, and let p_A denote the projection $A \times B \to A$. Then the 1-parameter group property may be written as the following identity[4] of maps $\mathbb{R}_1 \times \mathbb{R}_2 \times M \to M$ (we label the factors of $\mathbb{R}$ to keep track of order):

$$\Phi \circ (a \times \mathrm{id}_M) = \Phi \circ (p_{\mathbb{R}_1}, \Phi \circ p_{\mathbb{R}_2 \times M})$$

Then we simply differentiate and apply both sides to the vector field $\partial/\partial t_2$:

$$\Phi_*(\partial/\partial t) = \Phi_*(\Phi_*(\partial/\partial t)),$$

yielding $X = \Phi_* X$, as required. □

The fact that the diffeomorphisms φ_t preserve X automatically imply that they commute with the flows generated by X; this is no surprise, as we have $\varphi_t \varphi_{t'} = \varphi_{t+t'}$. Now we compute the way the flow of X acts on a different vector field Y.

Lemma 4.14. *Let Φ be the flow of X. If $[X, Y] = 0$ for a vector field Y, then $\Phi_* Y = Y$, i.e. $(\varphi_t)_* Y = Y$ for all t.*

Proof. Extend Φ to $\tilde{\Phi} = (p_{\mathbb{R}}, \Phi) : U \to \mathbb{R} \times M$. Then we have

$$\tilde{\Phi}_* \partial_t = \partial_t + X,$$

and also $\varphi_t = p_M \circ \tilde{\Phi} \circ j_t$. At $t = 0$, it is clear that $\tilde{\Phi}_* Y = Y$, since $\varphi_0 = \mathrm{id}_M$. So, we would like to compute $[\partial_t, \tilde{\Phi}_* Y]$, which measures the time derivative of $(\varphi_t)_* Y$:

$$[\partial_t, \tilde{\Phi}_* Y] = [\tilde{\Phi}_* \partial_t - X, \tilde{\Phi}_* Y],$$

but we may use the fact that $\tilde{\Phi}_* X = X$, from the previous Lemma. Hence we have

$$[\partial_t, \tilde{\Phi}_* Y] = \Phi_*[\partial_t, Y] - \Phi_*[X, Y], \tag{82}$$

where we have used the fact that diffeomorphisms preserve Lie brackets. Since Y is time-independent, the first term vanishes, and we obtain the result. □

Remark 4.15. Equation 82 has independent interest, as it expresses the Lie bracket of vector fields as the derivative of the action of the flow of one vector field on the other. To be precise, restricting Equation 82 to the $t = 0$ slice, we obtain

$$\tfrac{d}{dt}|_{t=0} (\varphi_t)_* Y = -[X, Y].$$

Finally, since φ_t preserves Y, it will commute with any flow generated by Y, yielding the following result.

Theorem 4.16. *If X, Y are vector fields generating flows φ_t, ψ_s, then $[X, Y] = 0$ if and only if $\varphi_t \psi_s = \psi_s \varphi_t$ for all s, t.*

[4] We also use the notation (f, g) for maps $f : A \to B$ and $g : A \to C$ to mean $\Delta \circ (f \times g)$, where $\Delta : A \to A \times A$ is the diagonal embedding.