Reading: Hatcher §1.1.

Exercise 1.

1. Determine the de Rham cohomology of $\mathbb{R}^3 \setminus Z$, where Z is the union of $\{x = y = 0, z \geq 0\}$, $\{y = z = 0, x \geq 0\}$, and $\{z = x = 0, y \geq 0\}$.
2. Use Mayer-Vietoris to determine the de Rham cohomology of a compact orientable surface of genus g.

Exercise 2. Hatcher §1.1, exercises 13, 16, 18.

Exercise 3. Let $p_0, p_1 \in S^2$ be distinct points. Show that any two paths $\gamma_0, \gamma_1 : [0,1] \to S^2 \; : \; \gamma_0(i) = \gamma_1(i) = p_i, \; i = 0, 1$ must be homotopic with fixed endpoints. Use this to describe the fundamental groupoid ΠS^2 completely. Also use this to compute $\pi_1(\mathbb{R}P^2)$. Compute $\pi_1(\mathbb{R}P^n)$.

Exercise 4. In $\mathbb{R}^3$, let C_1 be the z-axis, and let C_2 be the circle $\{x^2 + y^2 = 1 \text{ and } z = 0\}$. Compute $\pi_1(\mathbb{R}^3 \setminus \{C_1 \cup C_2\})$ and express it in terms of generators and relations. Draw a picture of the generators, and draw a picture of the relations. Use this to compute the fundamental group of $\mathbb{R}^3 \setminus \{\text{Hopf link}\}$.

Exercise 5. Let $\mathbf{I}$ be the category with two objects $\{0, 1\}$ and only one non-identity arrow $\iota : 0 \to 1$. If $\mathcal{C}, \mathcal{D}$ are categories, then a functor $F : \mathcal{C} \times \mathbf{I} \to \mathcal{D}$ is called a *natural transformation* from the functor $f_0 : \mathcal{C} \to \mathcal{D}$ to the functor $f_1 : \mathcal{C} \to \mathcal{D}$, where $f_i(X) = F(X, i), \; i = 0, 1$. Prove that there is a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$, and whose morphisms are natural transformations. What are the invertible morphisms (isomorphisms) in this category?

Two categories $\mathcal{C}, \mathcal{D}$ are *equivalent* when there are functors $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ such that $f \circ g$ is isomorphic to $\mathrm{id}_{\mathcal{D}}$ and $g \circ f$ is isomorphic to $\mathrm{id}_{\mathcal{C}}$, where isomorphism is in the sense above. Give an example of two categories which are equivalent but which have non-bijective objects (consider only small categories, i.e. categories whose objects and morphisms are each a set).

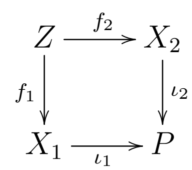
Exercise 6. A *coproduct* or *sum* of two objects X_1, X_2 in a category $\mathcal{C}$ is an object P, equipped with arrows $\iota_i : X_i \to P, \; i = 1, 2$ such that for any other object Q equipped with arrows $\nu_i : X_i \to Q$, there exists a unique arrow $\nu : P \to Q$ with $\nu_i = \nu \circ \iota_i$, for $i = 1, 2$. We draw the coproduct like this:

$$\begin{array}{ccc} & & X_2 \\ & & \big\downarrow {\scriptstyle \iota_2} \\ X_1 & \xrightarrow[\iota_1]{} & P \end{array}$$

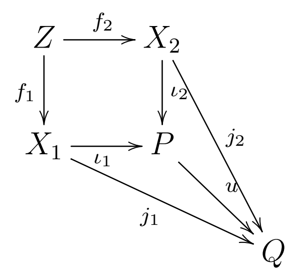
Show that if there are two coproducts of the pair X_1, X_2, then the two coproducts are canonically isomorphic.

Show that the category of sets, topological spaces, pointed topological spaces, groups, (and bonus: groupoids), always have coproducts, i.e. for any pair of objects X_1, X_2, there exists an object which is a coproduct of X_1, X_2.

Exercise 7. Given two objects X_1, X_2 in a category $\mathcal{C}$, and an object Z mapping to both X_i by $f_i : Z \to X_i$, the *fibered coproduct* (also called *fibered sum* or *pushout*) of X_1, X_2 over Z is an object P and two morphisms $\iota_i : X_i \to P$ such that the diagram commutes:



and such that (P, ι_1, ι_2) is universal for this diagram in the sense that for any other set (Q, j_1, j_2) fitting in the diagram, there must exist $u : P \to Q$ making the following diagram commute:



Show that the categories from the previous exercise always have fibered sums.