Generalized Complex Geometry
Lecture 5

October 14, 2015

Topics of this weeks seminar: Gauss-Manin, S^1-bundles, g-structures.

Definition 1. *A Fiber bundle $X \xrightarrow{\pi} S$ has a fiber F, a total space X and a smooth projection π which is surjective and locally trivial:*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi_U} & U \times F \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & C & \end{array} \tag{1}$$

where $U \subset S$ is an open neighborhood. This way we obtain gluing data

$$\begin{aligned} \Phi_V \circ \Phi_U^{-1} : \quad & (U \cap V) \times F \longrightarrow (U \cap V) \times F \\ & (x, p) \longmapsto (x, \Phi_{UV}(x)) \end{aligned} \tag{2}$$

where $\Phi_{UV}(x) : F \to F$ is a diffeomorphism.

This gives us a family of diffeomorphisms, parameterized by $x \in U \cap V$. We say that the Fiber bundle has the structure group $G \subset \mathrm{Diff}(F)$ when the gluing maps lie in G.

Example 2.

1. $F = \mathbb{C}^n$

$$\Phi_{UV}(x) \in GL_n \Rightarrow \textit{structure group } GL_n(\mathbb{C}) \tag{3}$$

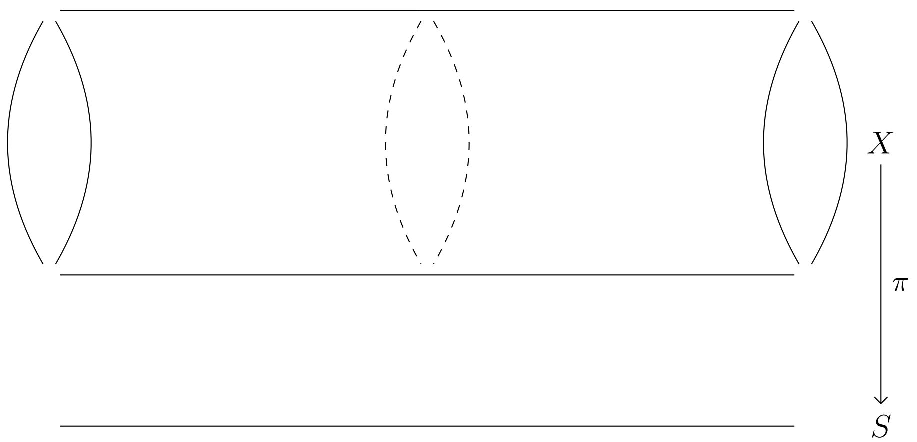
2. $F = \mathbb{R}^n$

$$\Phi_{UV}(x) \in \textit{Aff}(\mathbb{R}^n) \quad \textit{Affine Bundle} \tag{4}$$

3. $F = G$ *Lie group*

$$\Phi_{UV}(x) \textit{ lie in } G \curvearrowright G \textit{ by left multiplication} \tag{5}$$

Therefore X is a principal G-Bundle. Call it so because there is a residual right action of G on every fiber, making each fiber a principal homogeneous space or torsion of G.



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The Fiber bundle has a foliation

$$\begin{aligned} TX \supset F &= \ker(\pi_*) && \text{clearly involutive} \\ &= \mathrm{vert}(X) && \text{vertical tangent bundle} \\ &= T_{X/S} && \text{relative tangent bundle} \end{aligned} \tag{7}$$

$$0 \longrightarrow T_{X/S} \longrightarrow T_X \xrightarrow{\pi_*} \pi^* T_X \longrightarrow 0, \qquad \theta : T_X \dashrightarrow T_{X/S}, \quad \nabla : \pi^* T_X \dashrightarrow T_X \tag{8}$$

Definition 3. *A connection on the fiber bundle $X \xrightarrow{\pi} S$ is a choice of splitting of Eq. 8. The image $\nabla(\pi^* T_S) = H$ is called the horizontal distribution.*

Remark 4. *Splitting $\nabla \Rightarrow T_X = T_{X/S} \oplus H \Rightarrow$ proj. $\theta : T_X \to T_{X/S}$*
We can view the connection ∇ as $\theta \in \Omega^1_X(T_{X/S})$.

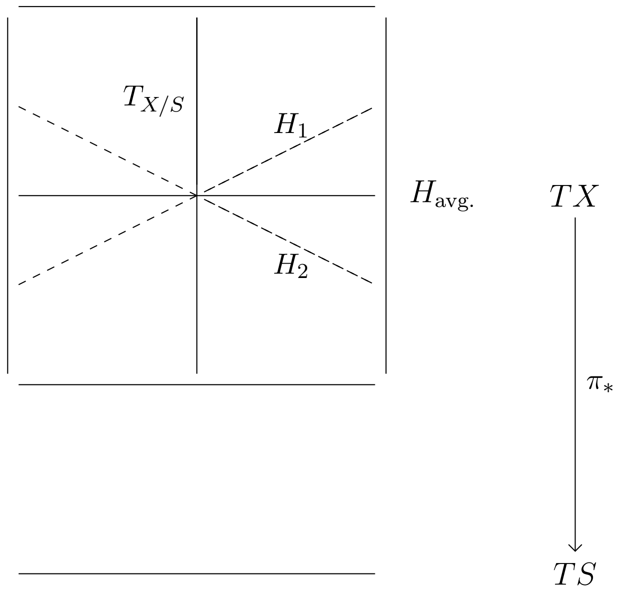
H may not be involutive, we can then measures the failure of being involutive.

Definition 5. *The Curvature of ∇ is for $X, Y \in C^\infty(X, \pi^* TS)$*

$$F(X,Y) = \theta\left([\nabla(X), \nabla(Y)]\right) \tag{9}$$

$$\Rightarrow F \in \Gamma\big(X, \Lambda^2 \pi^* T^* S \otimes T_{X/S}\big) = \Gamma\big(X, \Lambda^2 H^* \otimes V\big) \tag{10}$$

We study now the deRahm complex of the fiber bundle $X \xrightarrow{\pi} S$ by choosing a connection ∇. It is always possible to do this C^∞ since there exists a local splitting and splittings can be averaged.



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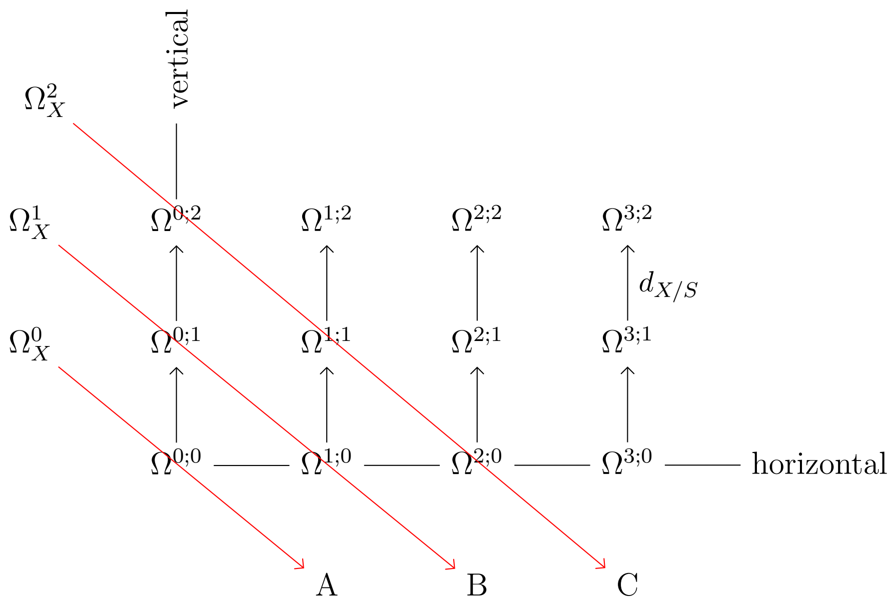
<u>The main point is:</u>

$$\begin{aligned} T_X &= V \oplus H \\ T_X^* &= V^* \oplus H^* \end{aligned} \tag{12}$$

$$\begin{aligned} \Lambda^k T_k^* &= \bigoplus_{p+q=k} \left(\Lambda^p V^* \otimes \Lambda^q H^* \right) \\ &= \bigoplus_{p+q=k} \Big(\underbrace{\Lambda^p T_{X/S}^* \otimes \Lambda^q \pi^* T_S^*}_{\text{a section of this is } \Omega^{q,p}} \Big) \end{aligned} \tag{13}$$

q is the horizontal and p is the vertical component.
<u>Key calculation:</u> How does $d : \Omega_X^k \to \Omega_X^{k+1}$ decompose into components?



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Let $\omega \in \Omega^{p;q}$.

$$d(w)(h_1, ..., h_p, v_1, ..., v_q) = d_{X/S}\omega \tag{15}$$

<u>Test the $\Omega^{p;q+1}$ component:</u>

1. vertical component:

$$\Omega^k \supset \Omega^{p;q} \xrightarrow{d} \Omega^{k+1} \xrightarrow{p} \Omega^{p;q+1}, \qquad \Omega^{p;q} \xrightarrow{d_{X/S}} \Omega^{p;q+1} \tag{16}$$

<u>Inspect for:</u>

(a) $p = 0$

$$\begin{array}{ccccccc} \Omega^{0;0} & \longrightarrow & \Omega^{0;1} & \longrightarrow & \Omega^{0;2} & \longrightarrow & ... \\ \| & & & & & & \\ C^\infty(X) & \xrightarrow[d_{X/S}]{} & \Omega^1_{X/S} & \xrightarrow[d_{X/S}]{} & \Omega^2_{X/S} & \longrightarrow & ... \end{array} \tag{17}$$

This is the foliated deRahm Complex (F, d_F).

(b) $p = 1$ (one horizontal leg)

$$\Gamma^\infty(H^*) \xrightarrow{\nabla^{\text{Bott}}} \Omega^1_{X/S}(H^*) \xrightarrow{d^{\nabla^{\text{Bott}}}} \Omega^2_{X/S}(H^*) \tag{18}$$

Bott connection on N_F

(c) $p = 2$ (two horizontal legs)

$$\Gamma^\infty(\Lambda^2 H^*) \xrightarrow{d^{\nabla^{\text{Bott}}}} \Omega^1_{X/S}(\Lambda^2 H^*) \xrightarrow{d^{\nabla^{\text{Bott}}}} \Omega^2_{X/S}(\Lambda^2 H^*) \tag{19}$$

induced from Bott connection

2. horizontal component:

$$\Omega^{p;q} \xrightarrow{d^\nabla} \Omega^{p+1;q} \tag{20}$$

this will depend on ∇.

3.

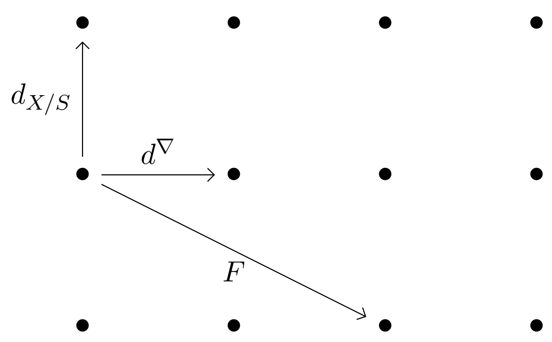
$$\begin{gathered} \Omega^{p;q} \xrightarrow{F} \Omega^{p+2;q-1} \\ F \in \Lambda^2 H^* \otimes V \end{gathered} \tag{21}$$

4. Observe that all other components vanish:

$$\begin{aligned} (d \underbrace{\omega}_{\in\Omega^{0;1}})(h_1, h_2) &= \underbrace{h_1\omega(h_2)}_{=0} - \underbrace{h_2\omega(h_1)}_{=0} - \omega([h_1, h_2]) \\ &= \omega(F(h_1, h_2)) \end{aligned} \tag{22}$$

$$(d\omega)(h_1, h_2, h_3) = 0$$

There are three parts of the derivative



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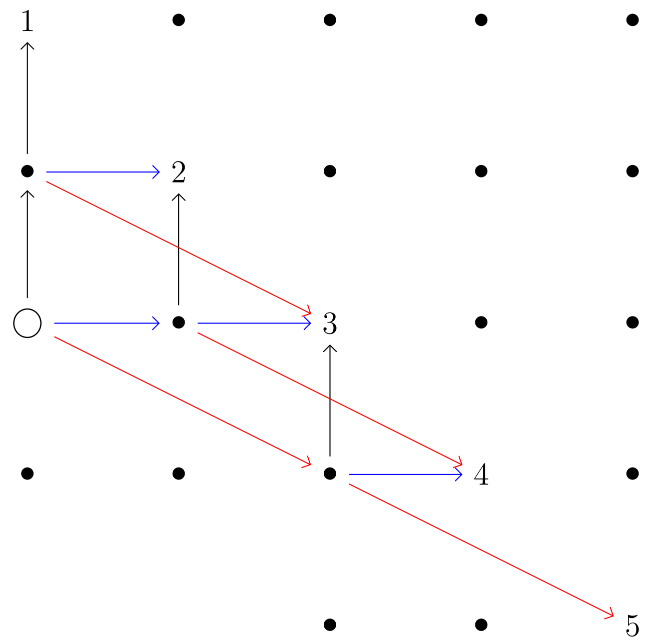
decomposition of d

We can consider the following 5 compositions.



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1. $d_{X/S}^2 = 0 \quad \Rightarrow \quad \begin{cases} \text{Foliated } d_F^2 = 0 \text{ (involutivity of F)} \\ \text{Bott connection is flat} \end{cases}$

2. $d_{X/S} d^{\nabla} + d^{\nabla} d_{X/S} = 0$

3. $\left(d^{\nabla}\right)^2 + F d_{X/S} + d_{X/S} F = 0$

4. $F d^{\nabla} + d^{\nabla} F = 0$

5. $FF = 0$

Implications from these equations:

1. There are vertical cohomology groupis (Foliated deRahm groups) (∞-dimensional due to the failure of $d_{X/S}$ being elliptic), but modules over $C^{\infty}(S)$.
   So $H^q_{X/S}(\pi^* \Omega^0_S)$ = sections of the vector bundle over S: $H^q_{X/S} \otimes \Lambda^p T^*_S$

2. $(-1)^k d^{\nabla}$ is a <u>chain map</u>. Therefore d^{∇} induces a map on the cohomology

$$
\begin{array}{ccc}
\Omega^2_{X/S} & \xrightarrow[+d^{\nabla}]{} & \Omega^2_{X/S}(H^*) \\
\uparrow d_{X/S} & & \uparrow d_{X/S} \\
\Omega^1_{X/S} & \xrightarrow[-d^{\nabla}]{} & \Omega^1_{X/S}(H^*) \\
\uparrow d_{X/S} & & \uparrow d_{X/S} \\
\Omega^0_{X/S} & \xrightarrow[+d^{\nabla}]{} & \Omega^0_{X/S}(H^*)
\end{array}
\tag{25}
$$

The Gauss-Manin connection (and it's natrual extension to Ω^k_S)

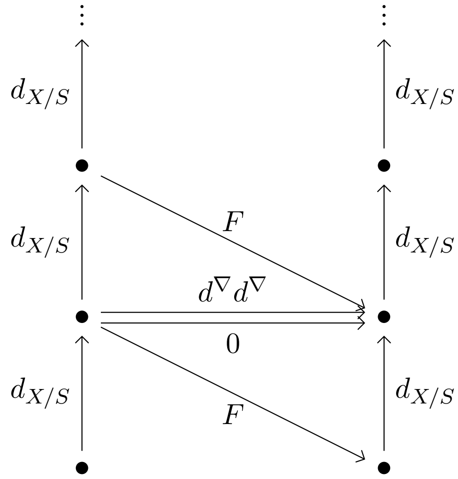
$$
H^q(\Omega^{\bullet}_{X/S}, d_{X/S}) = \Gamma\left(H^q_{X/S}\right) \xrightarrow{\left(d^{\nabla}\right)_*} \Gamma\left(H^q_{X/S} \otimes T^*_S\right)
\tag{26}
$$

This is a connection on the vector bundle $H^q_{X/S} \longrightarrow S$. The Leibniz rule follows from the Leibniz rule of d.

Remark 6.

$$
\begin{array}{l}
C^{\infty}(V) \xrightarrow{d^{\nabla}} C^{\infty}(V \otimes T^*_S) \xrightarrow{d^{\nabla}} C^{\infty}(V \otimes \Lambda^2 T^*_S) \xrightarrow{d^{\nabla}} \cdots \\
\quad \text{connection} \qquad \text{extended connection} \quad \cdots \\
\qquad \xrightarrow{d^{\nabla}} C^{\infty}(V \otimes \Lambda^n T^*_S) \xrightarrow{d^{\nabla}} 0
\end{array}
\tag{27}
$$

3. $d^\nabla \circ d^\nabla = 0$ in the foliated cohomology. Hence the Gauss-Manin connection is flat.



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F defines a chain homotopy between the chain maps $(d^\nabla)^2$ and 0. Therefore it is the same on $H^\bullet$.

Remark 7. *The Gauss-Manin connection is independent of the choice of* ∇*.*

Another approach to defining it without ∇:

$$\Omega_X^\bullet = (F_0^\bullet, d) \supset \underbrace{(F_1^\bullet, d)}_{\langle \Omega_S^1 \rangle} \supset (F_2^\bullet, d) \supset ... \supset (F_n^\bullet, d) \tag{29}$$

The deRahm complex on a fiber bundle is filtered. We have a Gauss-Manin connection on the vertical qth cohomology.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{p+1}^{p+q} \Big/ F_{p+2}^{p+q} & \longrightarrow & F_p^{p+q} \Big/ F_{p+2}^{p+q} & \longrightarrow & F_p^{p+q} \Big/ F_{p+1}^{p+q} & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & \Omega_S^{p+1}(\Omega_{X/S}^q, d_{X/S}) & \longrightarrow & \square & \longrightarrow & \Omega_S^p(\Omega_{X/S}^q, d_{X/S}) & & \end{array} \tag{30}$$

This is a short exact sequence (SES) of complexes. This gives us:

$$\begin{array}{ccccc}
\hookrightarrow \Omega_S^{p+1}(H^q_{X/S}) & \xrightarrow{d^\nabla} & \cdots & & \\
& & \overset{d^{GM}}{\longleftarrow\!\!\!\longleftarrow} & & \\
\hookrightarrow \Omega_S^{p+1}(H^{q-1}_{X/S}) & \xrightarrow{d^\nabla} & H^q(\square) & \xrightarrow{d^\nabla} & \Omega_S^p(H^q_{X/S}) \\
& & \overset{d^{GM}}{\longleftarrow\!\!\!\longleftarrow} & & \\
& & & & \cdots
\end{array} \tag{31}$$

Principal S^1-bundles:

$$\begin{array}{ccc}
S^1 \curvearrowright & P & \\
& \downarrow \pi & \\
& M &
\end{array}
\qquad \text{e.g.} \qquad
\begin{array}{ccc}
S^1 \curvearrowright & S^3 & \\
& \downarrow \pi & \\
& S^2 &
\end{array}$$

The gluing data for gluing $\{U_i\}$ of M is

$$\begin{aligned} g_{ij} : U_i \cap U_j \to U(1) \cong S^1 \\ g_{ij} g_{jk} g_{ki} = 1 \end{aligned} \tag{32}$$

The gluing data gives us a class $\{g_{ij}\} \in \check{H}^1\left(M, C^\infty_{S^1}\right)$ of smooth functions to S^1 on each $U_i \cap U_j$.

Locally constant integer functions:

$$\mathbb{Z} \longrightarrow C^\infty_{\mathbb{R}} \xrightarrow{e^{2\pi i \bullet}} C^\infty_{S^1} \tag{33}$$

$$\check{H}^1\left(C^\infty_{S^1}\right) \longrightarrow H^2_M(\mathbb{Z}) \tag{34}$$

Hence to each $\{g_{ij}\}$ S^1-principal bundle we can associate a class $\delta(g) = c_1(P) \in H^2(M, \mathbb{Z})$, the first Chern class of P. It classifies the principal S^1-bundle.

As a fiber bundle:

$$0 \longrightarrow T_{P/M} \longrightarrow TP \xrightarrow{\pi_*} \pi^*TM \longrightarrow 0 \tag{35}$$

Here $T_{P/M}$ has rank 1.
Since we have a S^1 action on P, $T_{P/M}$ has a global choice of basis ∂_θ, a vector field generator.

$$T_{P/M} \cong \underline{\mathbb{R}}\partial_\theta = \mathbb{R} \times P \longrightarrow P \tag{36}$$

$\underline{\mathbb{R}}\partial_\theta$ is the trivial line bundle generated by ∂_θ.

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow TP \xrightarrow{\pi_*} \pi^*TM \longrightarrow 0 \qquad (\theta: TP \dashrightarrow \underline{\mathbb{R}},\ \nabla: \pi^*TM \dashrightarrow TP) \tag{37}$$

These vector bundles are S^1-equivalent (the S^1 action in P extends to all of TP). Earlier we have studied TP by choosing the splitting from above.

Definition 8. *A Principal connection is an inveriant splitting of the sequence (preserving S^1 symmetry).*

The curvature of ∇ is

$$F(X,Y) = -\theta\left([\nabla X, \nabla Y]\right) \in \mathbb{R} \tag{38}$$

This gives us a function on P which is invariant under S^1, so it is also a function on M.

$$\Rightarrow F \in \Omega^2(M, \mathbb{R}) \qquad \text{i.e. } X, Y \in \mathfrak{X}(M) \tag{39}$$

$$\begin{aligned} [\nabla X, \nabla Y] &= \underbrace{[X,Y]}_{\text{horizontal part}} + \underbrace{F(X,Y)\partial_\theta}_{\text{vertical part}} \\ [X,Y] = \pi_* [\nabla X, \nabla Y] &= [\pi_* \nabla X, \pi_* \nabla Y] = [X,Y] \end{aligned} \tag{40}$$

Conclusion 9. *Once a principal connection ∇ is chosen*

$$TP = TM \oplus \underline{\mathbb{R}}\partial_\theta \tag{41}$$

and the bracket on S^1-invariant tangent vector fields is

$$[\nabla(X) + f\partial_\theta, \nabla(Y) + g\partial_\theta] = \nabla\left([X,Y]\right) + \left[(L_X g - L_Y f) + F(X,Y)\right]\partial_\theta \tag{42}$$

Refine this as follows: Quotient by S^1 and obtain an exact sequence of vector bundles on M.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{\mathbb{R}} & \longrightarrow & \overbrace{TP/S^1}^{dim(M)+1} & \xrightarrow{\pi_*} & \overbrace{\pi^*TM}^{rank=dim(M)} & \longrightarrow & 0 \\
 & & & & \| & \underset{\nabla}{\nwarrow\!\!\dashleftarrow\!\!\dashleftarrow} & & & \\
 & & & & A & & & &
\end{array}
\tag{43}
$$

Sections of these bundles on M are S^1-invariant sections on P. The principal connection is a splitting ∇ of the sequence over M (automatically S^1-invariant). The key object for studying S^1-invariant geometry of P is $A = {TP}/{S^1}$, called the Atiyah algebroid of P.

$$
0 \longrightarrow \underline{\mathbb{R}} \longrightarrow A \underset{\nabla}{\overset{\pi_*}{\rightleftarrows}} \pi^*TM \longrightarrow 0
\tag{44}
$$

The splitting ∇ implies that $A = TM \oplus \underline{\mathbb{R}}$, the curvature is $F \in \Omega^2(M, \mathbb{R})$ and the bracket on S^1-invariant tangent vector fields is

$$[X + f, Y + g] = [X, Y] + (L_X g - L_Y f) + F(X, Y) \tag{45}$$

Theorem 10. *The Atiyah algebroid inherits a Lie bracket on it's sections which is, upon a splitting ∇ with a curvature F, given by Eq. 45.*

In this way we get a family of Lie brackets

$$\Big(TMi \oplus \underline{\mathbb{R}}, [\ \ ,\ \ ]_F\Big) \tag{46}$$

Remark 11. *If this derives from* $\begin{array}{c} S^1 \to P \\ \downarrow \\ M \end{array}$*, then* $\frac{[F]}{2\pi} = c_1(P)$ *will be integral. Nevertheless, the construction works for any closed 2-form F.*

Setting:

$$\Big(M, F \in \Omega^2(M, \mathbb{R}), dF = 0\Big) \rightsquigarrow \Big(A = TM \oplus \underline{\mathbb{R}}, [\ \ ,\ \ ]_F\Big) \tag{47}$$

We can work with A as we would with a usual tangent bundle.

Example 12. *A has a deRahm complex*

$$\begin{aligned} A = TM \oplus \underline{\mathbb{R}} \quad &\Rightarrow \quad A^* = T^*M \oplus \underline{\mathbb{R}} \\ \Lambda^k A^* &= \Lambda^{k-1} T^*M \oplus \Lambda^k T^*M \end{aligned} \tag{48}$$

So a k-form $\varphi \in \Lambda^k A^*$ has the general form

$$\begin{array}{ccccc} \varphi & = & \theta \wedge \alpha & + & \beta \\ & & \pitchfork & & \pitchfork \\ & & V^* \wedge \Lambda^{k-1} H^* & & \Lambda^k H^* \end{array} \tag{49}$$

where $\theta \in \Omega^1(P)$ is the connection 1-form.
The differential operator is:

$$d_A : \Gamma(\Lambda^k A^*) \longrightarrow \Gamma(\Lambda^{k+1} A^*) d\varphi = d\theta \wedge \alpha - \theta \wedge d\alpha + d\beta \tag{50}$$

$$\begin{array}{ccc} \Omega_M^{k-1} & \xrightarrow{\quad -d_M \quad} & \Omega_A^k \\ & \searrow^{F} & \\ \oplus & & \oplus \\ \Omega_M^k & \xrightarrow{\quad d_M \quad} & \Omega_M^{k+1} \\ \| & & \| \\ \Omega_M^k & \xrightarrow{\quad d_A \quad} & \Omega_A^{k+1} \end{array} \tag{51}$$

So

$$d_A = \begin{pmatrix} -d_M & 0 \\ F & d_M \end{pmatrix} \tag{52}$$

<u>What structures could we now study?</u>

Example 13. *Let Σ be a 2-manifold and $\omega \in \Omega^2_\Sigma$ be a closed 2-form $d\omega = 0$, then*

$$A = T\Sigma \oplus \mathbb{R} \tag{53}$$

has rank 3.
<u>*One can study*</u>

1. *co-rank 1 distributions, e.g. co-rank 1 subbundles $F \subset A$*

2. *involutivity $[F, F]_\omega \subset F$*

As before we can encode such distributions in

$$\mu = \mu_0 + \mu_1 \tag{54}$$

where $\mu \in \Omega^1_A = \Omega^0_M \oplus \Omega^1_M$, *such that*

$$\begin{aligned} \mu \wedge d\mu &= 0 \\ \Leftrightarrow (\mu_0 + \mu_1) \wedge d(\mu_0 + \mu_1) &= 0 \end{aligned} \tag{55}$$

simplyfy: $\omega = 0$

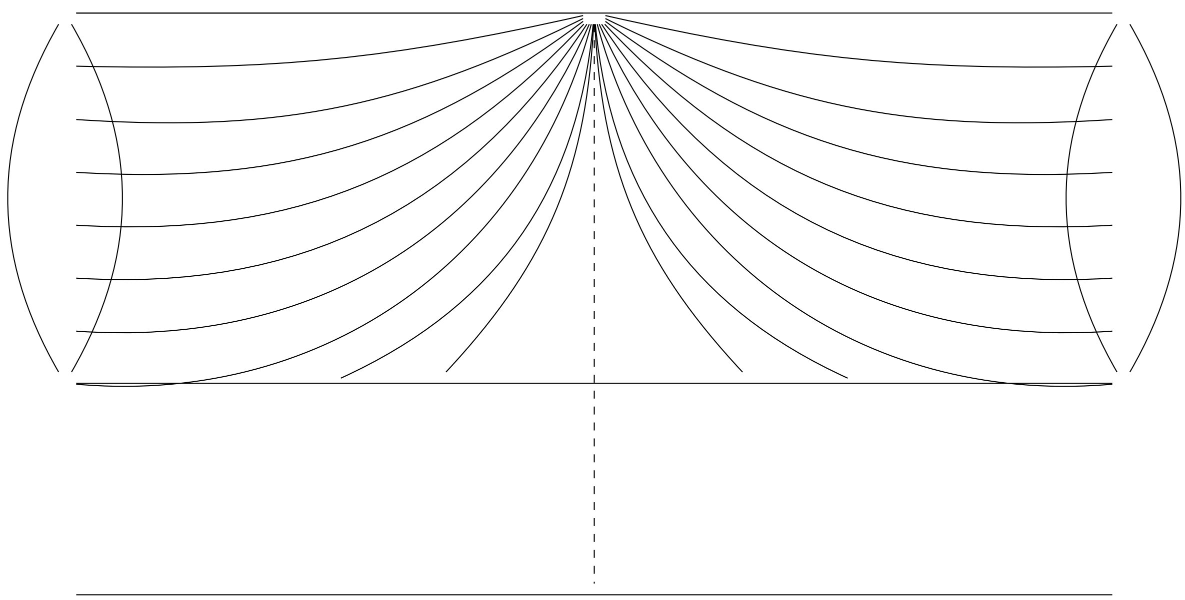
$$\begin{aligned} \Leftrightarrow \mu_0 \wedge d\mu_1 + \mu_1 \wedge d\mu_0 + \underbrace{\mu_1 \wedge d\mu_1}_{dim=2 \ \Rightarrow \ =0} &= 0 \\ \Leftrightarrow d(\mu_0 \wedge \mu_1) &= 0 \end{aligned} \tag{56}$$

Require that $\mu \neq 0$ *anywhere.* $\Leftrightarrow \mu_0 + \mu_1 \neq 0$ *anywhere.* $\mu_0 \wedge \mu_1$ *is a closed 1-form on* Σ, *but it could have zeros with violating the previous statement.*

Example 14. *Consider* $(\mathbb{R}^2, x, y)$ *and* $\mu = f(x) + dx$. *For example* $\mu = x + dx$. *Then clearly* $d(\mu_0 \wedge \mu_1) = 0$.
"Upstairs" we have

$$\mu = x \wedge d\theta + dx \tag{57}$$

This is a usual foliation on $S^1 \times \mathbb{R}^2_{x,y}$ *as* $\mu \wedge d\mu = 0$.



$x = 0$

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Problem 15. *Find an interesting genetic foliation on a surface* Σ*, for example one with* $gGV \neq 0$.