GENERALIZED GEOMETRY: LECTURE 3

1. Derivations of $\Omega^\bullet(M)$.

Given a manifold M, consider the algebra of differential forms $(\Omega^\bullet(M), \wedge, d)$. Note that $\Omega^\bullet(M)$ is $\mathbb{Z}$-graded, and $\wedge$ is an associative, graded commutative product. Furthermore the de Rham differential d is a degree $+1$ derivation of the algebra satisfying $d \circ d = 0$, or equivalently $[d, d] = 0$ (this property is sometimes expressed by saying that d is a *homological vector field*). Therefore, the algebra of differential forms is a *commutative differential graded algebra* (cdga).

One consequence of the fact that $d^2 = 0$ is that we get a cochain complex:

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1} \xrightarrow{d} \Omega^k \xrightarrow{d} \Omega^{k+1} \xrightarrow{d} \cdots$$

and can therefore define de Rham cohomology groups (actually vector spaces)

$$H^k_{dR}(M) = \frac{\ker(d|_{\Omega^k})}{\operatorname{im}(d|_{\Omega^{k-1}})}.$$

The wedge product $\wedge$ descends to a well-defined *cup product* on the level of cohomology, which we denote $\cap$; this makes $(H^\bullet(M), \cap)$ into a cga.

In the case that M is compact $H^\bullet(M)$ is finite-dimensional. This follows from the fact that $(\Omega^\bullet(M), d)$ is an *elliptic complex*.

1.1. **An aside on differential operators.** Let E and F be vector bundles over M; their spaces of sections, $\Gamma(E), \Gamma(F)$, are $C^\infty(M, \mathbb{R})$-modules. Given a function $f \in C^\infty(M, \mathbb{R})$, let $m_f : \Gamma(E) \to \Gamma(E)$ to be the module endomorphism defined by $m_f(s) = fs$ for $s \in \Gamma(E)$.

Definition. Suppose

$$D : \Gamma(E) \to \Gamma(F)$$

is $\mathbb{R}$-linear.

(i) D is a *differential operator of order* 0 if

$$[D, m_f] = 0, \text{ for all } f \in C^\infty(M, \mathbb{R}).$$

In other words, given $s \in \Gamma(E)$, $D(fs) = f(Ds)$, implying that D is a module homomorphism (i.e. D is a vector bundle morphism $E \to F$). So

$$\operatorname{Diff}^0(E, F) = \operatorname{Hom}_{VB}(E, F) = \Gamma(E^* \otimes F),$$

where $\mathrm{Diff}^0(E,F)$ denotes the $C^\infty(M,\mathbb{R})$-module of differential operators of order 0.

(ii) D is a *differential operator of order* ≤ 1 if

$$[D, m_f] \in \mathrm{Diff}^0(E,F), \text{ for all } f \in C^\infty(M,\mathbb{R}),$$

or equivalently

$$[[D, m_f], m_g] = 0, \text{ for all } f, g \in C^\infty(M,\mathbb{R}).$$

Let $\mathrm{Diff}^{\leq 1}(E,F)$ denote the $C^\infty(M,\mathbb{R})$-module of differential operators of order ≤ 1.

(iii) We inductively define the $C^\infty(M,\mathbb{R})$-module of *differential operators of order* $\leq k$, $\mathrm{Diff}^{\leq k}(E,F)$, to be the collection of $\mathbb{R}$-linear morphisms D satisfying

$$[D, m_f] \in \mathrm{Diff}^{\leq k-1}(E,F), \text{ for all } f \in C^\infty(M,\mathbb{R}).$$

Remark. We get a *filtration* of modules

$$\mathrm{Diff}^0 \subseteq ... \subseteq \mathrm{Diff}^{\leq k-1} \subseteq \mathrm{Diff}^{\leq k} \subseteq ...$$

Suppose $D \in \mathrm{Diff}^{\leq 1}(E,F)$, and $f \in C^\infty(M,\mathbb{R})$. Then as we saw $[D, m_f]$ is a section of $E^* \otimes F$. Given D, we can define a map

$$\psi : C^\infty(M) \to \Gamma(E^* \otimes F), \ f \mapsto [D, m_f],$$

which is clearly $\mathbb{R}$-linear. Furthermore, ψ is a derivation:

$$\begin{aligned} \psi(fg) = [D, m_{fg}] &= D m_f m_g - m_f m_g D \\ &= D m_f m_g - m_f D m_g + m_f D m_g - m_f m_g D \\ &= [D, m_f] m_g + m_f [D, m_g] \\ &= g\psi(f) + f\psi(g). \end{aligned}$$

Therefore, the map ψ is of the form

$$\psi(f) = \langle \sigma(D), df \rangle,$$

for $\sigma(D) \in \Gamma(T \otimes E^* \otimes F)$, the *principal symbol* of D. Note that we can write down a local expression for D as follows

$$D = \sum_i a_i \partial_{x_i} + b,$$

and $\sum_i a_i \partial_{x_i}$ is the principal symbol.

Remark. For $D \in \mathrm{Diff}^{\leq k}(E,F)$, we can still define the principal symbol $\sigma(D)$, which will now be a section of $S^k T \otimes E^* \otimes F$.

1.2. **The de Rham complex is an elliptic complex.** Returning to the de Rham complex, we note that

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

is a differential operator of order 1:

$$d|_{\Omega^k} \in \mathrm{Diff}^{\leq 1}(\bigwedge^k T^*, \bigwedge^{k+1} T^*).$$

In order to see this, we need to check that for all $f \in C^\infty(M, \mathbb{R})$ the commutator $[d, m_f]$ lies in $\mathrm{Hom}_{VB}(\bigwedge^k T^*, \bigwedge^{k+1} T^*)$: given $\rho \in \Omega^k$

$$[d, m_f]\rho = d(f\rho) - f d\rho = df \wedge \rho.$$

This verifies the claim since $[d, m_f] = df\wedge$ is indeed a homomorphism of bundles. Now what is the principal symbol of d? First of all

$$\begin{aligned} \sigma(d) &\in \Gamma(T \otimes \mathrm{Hom}(\bigwedge^k T^*, \bigwedge^{k+1} T^*)) \\ &= \mathrm{Hom}(T^*, \mathrm{Hom}(\bigwedge^k T^*, \bigwedge^{k+1} T^*)). \end{aligned}$$

And so given $\xi \in \Gamma(T^*)$,

$$\langle \sigma(d), \xi \rangle : \bigwedge^k T^* \to \bigwedge^{k+1} T^*, \ \rho \mapsto \xi \wedge \rho.$$

Hence from $(\Omega^\bullet, d)$ we get a *symbol sequence* for every $\xi \in \Gamma(T^*)$

$$\cdots \longrightarrow \bigwedge^{k-1} T^* \xrightarrow{\xi\wedge} \bigwedge^k T^* \xrightarrow{\xi\wedge} \bigwedge^{k+1} T^* \longrightarrow \cdots$$

For $\xi \neq 0$, this sequence is also known as the *Koszul sequence*.

Proposition. Given $a \in V$, a nonzero element of a finite dimensional vector space (of dimension n), the Koszul sequence

$$0 \longrightarrow \bigwedge^0 V \xrightarrow{a\wedge} \bigwedge^1 V \xrightarrow{a\wedge} \bigwedge^2 V \xrightarrow{a\wedge} \cdots \longrightarrow \bigwedge^n V \longrightarrow 0$$

is exact. That is to say $\mathrm{im}(a\wedge) = \ker(a\wedge)$.
Proof. The proof relies on the observation that for nonzero a, and $\rho \in \bigwedge^k V$, $a \wedge \rho = 0$ if and only if $\rho = a \wedge \eta$ for some $\eta \in \bigwedge^{k-1} V$. $\square$

The above conclusion applies fibre-wise to the symbol sequence of the de Rham complex, and we therefore say that the symbol sequence is exact. A differential complex with exact symbol sequence is said to be *elliptic*. The fact that the de Rham complex of a compact manifold is elliptic implies that its cohomology is finite dimensional.

1.3. **Derivations et al. of** $(\Omega^\bullet(M), \wedge)$**.** Let us now describe some derivations of the cga $(\Omega^\bullet(M), \wedge)$. We have already seen that d is a derivation of degree 1:

$$d \in \mathrm{Der}^{+1}(\Omega^\bullet).$$

Given any vector field, we can define a derivation of degree -1.

Definition. Let $X \in \Gamma(T) = \mathfrak{X}^1(M)$ be a vector field on M. The *interior product*

$$i_X : \Omega^k \to \Omega^{k-1}$$

is the unique degree -1 derivation of $(\Omega^\bullet(M), \wedge)$ such that for all $f \in C^\infty(M, \mathbb{R})$

(i) $i_X f = 0$,

(ii) $i_X(df) = df(X) = X(f)$.

To compute its action on the rest of $(\Omega^\bullet(M), \wedge)$ we extend by the derivation rule:

$$i_X(\alpha \wedge \beta) = i_X(\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge i_X(\beta).$$

Remark. It is also possible to give an invariant definition of i_X as follows. Let $\rho \in \Omega^{k+1}$, then

$$(i_X \rho)(X_1, ..., X_k) = \rho(X, X_1, ..., X_k).$$

We note the following proposition which follows from the invariant definition.

Proposition. $i_X^2 = 0$.

Because the derivations of $(\Omega^\bullet(M), \wedge)$ form a graded Lie algebra (gLa): $\mathrm{Der}^\bullet(\Omega^\bullet)$, given any two derivations, we can use the Lie bracket on $\mathrm{Der}^\bullet(\Omega^\bullet)$ to produce a third one. As such, the derivations d and i_X can be used to generate a new derivation of degree 0

$$L_X := [i_X, d] = i_X d - (-1)^1 d i_X = i_X d + d i_X \in \mathrm{Der}^0(\Omega^\bullet).$$

Cartan's magic formula identifies L_X with the usual Lie derivative. That is to say, given $\rho \in \Omega^k$,

$$L_X \rho = \left.\frac{d}{dt}\right|_{t=0} (\varphi^X_{-t})^* \rho,$$

where φ^X_t is the flow of X. A simple calculation shows that $[i_X, L_X] = [L_X, d] = 0$, so these derivations form a closed system.

Given two vector fields $X, Y \in \mathfrak{X}^1$, let us try to combine the various derivations they produce. First of all, $[i_X, i_Y] = 0$ since it is a derivation of degree -2, and $(\Omega^\bullet(M), \wedge)$ is generated in degrees 0 and 1. Next, we have the following identity which is due to Cartan.

Proposition. **(Cartan's Formula)** $[L_X, i_Y] = i_{[X,Y]_{\mathbf{Lie}}}$, where $[X,Y]_{\mathbf{Lie}}$ denotes the Lie bracket of vector fields.
Proof. We only need to check that the two derivations agree on functions and their differentials.

$$[L_X, i_Y]f = 0,$$

since it is a derivation of degree -1. And

$$\begin{aligned} [L_X, i_Y]df &= L_X Y(f) - i_Y L_X df = i_X d(Yf) - i_Y d(i_X df) \\ &= XYf - YXf = [X,Y]_{\mathbf{Lie}} f. \ \square \end{aligned}$$

Finally, combining the above identities, we get $[L_X, L_Y] = L_{[X,Y]_{\mathbf{Lie}}}$.

1.4. **Aside on derived brackets.** It turns out that the derivations of degree -1 are equivalent to vector fields on M. To see this note that a derivation $D \in \mathrm{Der}^{-1}(\Omega^\bullet)$ has the following natural action on $C^\infty(M)$:

$$f \mapsto D(df)$$

which is a derivation since

$$D(d(fg)) = D(fdg + gdf) = fD(dg) + gD(df).$$

Therefore, D defines a vector field X on M such that $i_X = D$. As a result, the Cartan formula above can actually be thought of as a definition of the Lie bracket for vector fields. Given vector fields X and Y, we can *define* $[X,Y]_{\mathbf{Lie}}$ to be the vector field induced by $[L_X, i_Y] = [[i_X, d], i_Y]$. This is an example of the *derived bracket construction*, in which given a differential graded Lie algebra $(L, [\ ,\ ], D)$, we define the *derived bracket* on L to be the bilinear map:

$$L \times L \ni (a,b) \mapsto [Da, b].$$

In the present case, the differential in question is given by bracketing with d, i.e. $a \mapsto [a, d]$. This shows that there is a kind of duality between the de Rham differential d and the Lie bracket on vector fields: just as d can be defined in terms of the Lie bracket:

$$\begin{aligned} d\omega(X_0, ..., X_k) = \sum_i (-1)^i X_i \omega(X_0, ..., \hat{X}_i, ..., X_k) \\ + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j]_{\mathbf{Lie}}, ..., \hat{X}_i, ..., \hat{X}_j, ...), \end{aligned}$$

so too can the Lie bracket be defined in terms of the de Rham differential via Cartan's formula.

1.5. **The Schouten bracket.** The Lie bracket on $\mathfrak{X}^1$ can be extended to a bracket on multivector fields $\mathfrak{X}^\bullet(M) = \oplus_{k\geq 0}\Gamma(\bigwedge^k T)$.

Definition. The *Schouten bracket* on multivector fields is defined to be the bilinear map

$$[\ ,\ ] : \mathfrak{X}^{p+1} \times \mathfrak{X}^{q+1} \to \mathfrak{X}^{p+q+1}$$

satisfying

(i) $[f, g] = 0$ and $[X, f] = X(f)$ for functions f, g and vector field X,

(ii)

$$[X_1 \wedge ... \wedge X_k, Y_1 \wedge ... \wedge Y_l] = \sum_{i,j}(-1)^{i+j}[X_i, Y_j]_{\mathbf{Lie}} \wedge X_1 \wedge ... \wedge \hat{X}_i \wedge ... \wedge X_k \wedge Y_1 \wedge ... \wedge \hat{Y}_j \wedge ... \wedge Y_l$$

where X_i and Y_j are vector fields.

Example.

$$\begin{aligned}[X\wedge Y, X \wedge Y] &= [X, X] \wedge Y \wedge Y - [X, Y] \wedge Y \wedge X - [Y, X] \wedge X \wedge Y + [Y, Y] \wedge X \wedge X \\ &= 2[X, Y] \wedge X \wedge Y.\end{aligned}$$

Notice that this vanishes precisely when $[X, Y]$ is in the span of X and Y, or equivalently, when $\langle X, Y\rangle$ is involutive.

Note that the Schouten bracket cannot be a graded Lie bracket since it does not respect the degrees of multivector fields. However, if we shift the grading on $\mathfrak{X}^\bullet$ to $\mathfrak{X}^{\bullet+1}$ then the Schouten bracket does respect this new grading. In this grading, smooth functions lie in degree -1 and vector fields lie in degree 0. With respect to this grading, the Schouten bracket becomes a graded Lie bracket on multivector fields, meaning that it is

(i) graded antisymmetric:

$$[P, Q] = -(-1)^{pq}[Q, P], \quad \text{for } P \in \mathfrak{X}^{p+1}, Q \in \mathfrak{X}^{q+1},$$

(ii) and satisfies the graded Jacobi identity:

$$(-1)^{pr}[P, [Q, R]] + (-1)^{qp}[Q, [R, P]] + (-1)^{rq}[R, [P, Q]] = 0,$$

for $P \in \mathfrak{X}^{p+1}, Q \in \mathfrak{X}^{q+1}, R \in \mathfrak{X}^{r+1}$.

Multivector fields with the original grading and the wedge product $(\mathfrak{X}^\bullet, \wedge)$ form a cga, and the Schouten bracket satisfies a Liebniz rule involving both gradings on multivector fields:

$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{pq} Q \wedge [P, R]$$

where $P \in \mathfrak{X}^{p+1}$ and $Q \in \mathfrak{X}^q$. This fact can be expressed by saying that the adjoint action of multivector fields

$$P \mapsto ad_P = [P,\ ]$$

gives a homomorphism of graded Lie algebras from $(\mathfrak{X}^{\bullet+1}, [\ ,\ ])$ to the algebra of derivations with respect to the unshifted grading $\mathrm{Der}^\bullet(\mathfrak{X}^\bullet, \wedge)$.

Remark. $(\mathfrak{X}^{\bullet(+1)}, [\ ,\ ], \wedge)$ is an example of a *Gerstenhaber algebra*.

Just as the usual Lie bracket turned out to be a derived bracket, so too is the Schouten bracket.

Proposition. Let $P \in \mathfrak{X}^{p+1}, Q \in \mathfrak{X}^{q+1}$, then $[P, Q]$ is a derived bracket.
Proof. A multivector $P = X_1 \wedge ... \wedge X_{p+1}$ acts on a differential form $\rho \in \Omega^\bullet$ by interior multiplication, i_P, defined by

$$i_P(\rho) = i_{X_{p+1}} i_{X_p} ... i_{X_2} i_{X_1} \rho.$$

Note that this is not a derivation. Then

$$i_{[P,Q]} = (-1)^{pq}[[i_P, d], i_Q],$$

where the brackets on the right are commutators of operators on the de Rham complex. □

We end the discussion of Schouten brackets with an application to Poisson geometry. Let $\Pi \in \mathfrak{X}^2$ be a bivector field. This determines a bracket on smooth functions as follows:

$$\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M), \quad (f, g) \mapsto i_\Pi(df \wedge dg).$$

This bracket is clearly $\mathbb{R}$-bilinear and antisymmetric. Furthermore, this bracket satisfies a Leibniz rule:

$$\{fg, h\} = i_\Pi(d(fg) \wedge dh) = i_\Pi(fdg \wedge dh + gdf \wedge dh) = f\{g, h\} + g\{f, h\}.$$

Therefore, this defines a Poisson bracket if and only if it satisfies the Jacobi identity. This condition can be stated in terms of the Schouten bracket.

Proposition. $\{\ ,\ \}$ is a Poisson bracket if and only if $[\Pi, \Pi] = 0$.
Proof. First of all,

$$[\Pi, \Pi] = 0 \iff i_{[\Pi,\Pi]}(df \wedge dg \wedge dh) = 0 \text{ for all } f, g, h \in C^\infty(M),$$

and so we need only show that the right hand side is equivalent to the Jacobi identity. To do this we will make use of the derived bracket formulation of the Schouten bracket.

$$\begin{aligned} i_{[\Pi,\Pi]}(df \wedge dg \wedge dh) &= -[[i_\Pi, d], i_\Pi] df \wedge dg \wedge dh \\ &= -(i_\Pi d i_\Pi - d i_\Pi i_\Pi - i_\Pi i_\Pi d + i_\Pi d i_\Pi) df \wedge dg \wedge dh \\ &= -2 i_\Pi d i_\Pi (df \wedge dg \wedge dh) \\ &= -2 i_\Pi d(\{f,g\}dh + \{g,h\}df + \{h,f\}dg) \\ &= -2(\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}). \ \square \end{aligned}$$

1.6. **Back to derivations of** $(\Omega^\bullet(M), \wedge)$**.** Let $X \in \mathfrak{X}^1$ and $\omega \in \Omega^k$. We can define the *interior product* by $K = \omega \otimes X$ as follows:

$$i_K : \Omega^p \to \Omega^{p+k-1}, \ \rho \mapsto \omega \wedge i_X(\rho).$$

As can be checked, this defines a derivation of degree $k-1$. And by extending linearly we can define the interior product by any element in $\Omega^k(T) = \Gamma(\bigwedge^k T^* \otimes T)$. This defines a morphism of modules

$$i : \Omega^k(T) \to \mathrm{Der}^{k-1}(\Omega^\bullet),$$

and hence a collection of new derivations of the de Rham complex.

Note that when $k = 1$, $\Omega^1(T) = \Gamma(T^* \otimes T) = \mathrm{End}(T)$, which has a tensorial Lie bracket, namely, the commutator bracket. This can be generalized to a tensorial graded Lie bracket on $\Omega^{\bullet+1}(T)$, known as the *Nijenhuis-Richardson* bracket:

$$\begin{aligned} [\ ,\ ]^\wedge : \Omega^{p+1}(T) \times \Omega^{q+1}(T) &\to \Omega^{p+q+1}(T), \\ (K, L) &\mapsto i_K L - (-1)^{pq} i_L K. \end{aligned}$$

This bracket satisfies the property that $[i_K, i_L] = i_{[K,L]^\wedge}$.

Note that when $k = 0$, $\Omega^0(T) = \Gamma(T) = \mathfrak{X}^1$, which has the Lie bracket. This can be generalized to a graded Lie bracket on $\Omega^\bullet(T)$, known as the *Frölicher-Nijenhuis* bracket. To define this, we first define the Lie derivative along $K \in \Omega^k(T)$ as follows:

$$L_K = [i_K, d].$$

This is a degree k derivation. Then the Frölicher-Nijenhuis bracket of $K \in \Omega^k(T)$, and $L \in \Omega^l(T)$ is the unique element $[K, L] \in \Omega^{k+l}(T)$ such that

$$[L_K, L_L] = L_{[K,L]}.$$

Note that being an extension of the usual Lie bracket, the Frölicher-Nijenhuis bracket involves taking derivatives. This bracket satisfies the following identity, analogous to Cartan's formula:

$$[L_K, i_L] = i_{[K,L]} - (-1)^{kl} L_{i_L K}.$$

Furthermore, the adjoint action with respect to this bracket defines a morphism of graded Lie algebras

$$ad : (\Omega^\bullet(T), [\ ,\ ]) \to \mathrm{Der}^\bullet(\Omega^\bullet, \wedge).$$

Remark. The Frölicher-Nijenhuis bracket is important in complex geometry as it is involved in checking that an almost complex structure is integrable (and hence defines a complex structure). If $J \in \Omega^1(T)$ is an almost complex structure, then $[J, J] \in \Omega^2(T)$ is the *Nijenhuis tensor* of J. The Newlander-Nirenberg theorem states that an almost complex structure J is integrable if and only if $[J, J] = 0$.

2. Foliations

Let M be a smooth manifold. Recall that a *distribution* D is a subbundle of the tangent bundle of M, and that a distribution is *involutive* if $[D, D] \subseteq D$, meaning that given sections $X, Y \in \Gamma(D)$, we have that $[X, Y] \in \Gamma(D)$. For a simple example of a distribution of rank k consider $\mathbb{R}^n$, with D given by the span of the first k coordinate vector fields: $\partial_{x_1}, ..., \partial_{x_k}$. The Frobenius theorem says that locally all involutive distributions look like this example. In the rank 1 case this can be stated as follows:

Proposition. If X is a vector field which is non-zero at $p \in M$, then there exist coordinates $x_1, ..., x_n$ such that $X = \partial_{x_1}$ near p.
Proof. First choose coordinates $y_1, ..., y_n$ centred at p with $X_p(y_1) \neq 0$. As such the submanifold defined by $y_1 = 0$ is transverse to X near p. Denote this codimension 1 submanifold by N. There exists a neighbourhood U of p such that for all $q \in U$ there is a unique point $\bar{q} \in N$ such that the flow of X takes $\bar{q}$ to q. Let $x_1(q)$ denote the time of flow from $\bar{q}$ to q, and for $i > 1$, let $x_i(q) = y_i(\bar{q})$. These give coordinates on U. Furthermore, by construction, the integral curves of X have the form $\gamma(t) = (t, x_2, ..., x_n)$ and so in these coordinates $X = \partial_{x_1}$ as required. $\square$

Generalizing this to k vectors, we get the following theorem of Frobenius:

Theorem. (Frobenius) Let $X_1, ..., X_m$ be vector fields on M which are linearly independent at a point $p \in M$, and suppose that $[X_i, X_j] \in \mathrm{span}(X_1, ..., X_m)$ for all i, j. Then there exist coordinates $x_1, ..., x_m, x_{m+1}, ..., x_n$ such that

$$\mathrm{span}(X_1, ..., X_m) = \mathrm{span}(\partial_{x_1}, ..., \partial_{x_m}).$$

Remark. These coordinates give a *foliation chart* for the distribution spanned by the X_i; they put the distribution in a standard local form.
Proof. The key step in the proof is to construct a collection of m linearly independent *commuting* vector fields with the same span as the X_i. Once this is achieved, the proof reduces to a generalized version of the above proposition. We sketch the main idea of the construction of these commuting vector fields in the case that $m = 2$.

Let X and Y be two linearly independent vector fields. By the above proposition we can find coordinates such that

$$X = \partial_1, \quad Y = \sum_{i=1}^{n} a_i \partial_i.$$

Without loss of generality, we can assume that $a_1 = 0$ (subtract $a_1 X$ from Y does not affect the span). Then

$$[X, Y] = \sum_{i=2}^{n} (\partial_1 a_i)\partial_i.$$

By assumption $[X, Y] \in \mathrm{span}(X, Y)$ but cannot have any component along X. Hence $[X, Y] = \lambda Y$. We want to modify Y so that $\lambda = 0$, so let us replace Y by $e^{\phi} Y$ and solve the following equation

$$0 = [X, e^{\phi}Y] = X(e^{\phi})Y + e^{\phi}[X, Y] = (X(\phi) + \lambda)e^{\phi}Y.$$

Therefore we need $X(\phi) + \lambda = 0$. This is solved by letting

$$\phi(x_1, ..., x_n) = \int_0^{x_1} -\lambda(t, x_2, ..., x_n)dt. \quad \square$$

2.1. **Equivalent formulation of Frobenius.** The Frobenius theorem essentially says that distributions have foliation charts if and only if they are involutive. There are two main ways of describing involutive distributions. The first is the description we saw above, namely, a distribution F is involutive if and only if $[F, F] \subseteq F$. The second description involves differential forms. If F is a distribution of corank k, then it is locally defined by k 1-forms $\theta_1, ..., \theta_k \in \Omega^1(M)$ (these give a basis for the annihilator of F). These generate an ideal $\mathcal{I} \subseteq (\Omega^{\bullet}, \wedge)$ (this is the ideal of all forms annihilating F). Then we have

Proposition. F is involutive if and only if $d\mathcal{I} \subseteq \mathcal{I}$ (i.e. $\mathcal{I}$ is a *differential ideal*).

Example Let $k = 1$, so that $\mathcal{I}$ is locally generated by a single non-vanishing 1-form θ. Involutivity requires that $d\theta \in \langle\theta\rangle$, so $d\theta = \theta \wedge \alpha$ for some 1-form α, or equivalently $\theta \wedge d\theta = 0$.

Proof. We give the proof in the case $k = 1$. Integrability means that if $i_X\theta = i_Y\theta = 0$, then $i_{[X,Y]}\theta = 0$. But

$$i_{[X,Y]}\theta = [L_X, i_Y]\theta = -i_Y i_X d\theta.$$

Now we can locally decompose $T^* = \langle\theta\rangle \oplus C$ so that $\Omega^2 = (\langle\theta\rangle \otimes C) \oplus \wedge^2 C$. Then if X and Y kill θ, they must lie in C^*, and hence $X \wedge Y \in \wedge^2 C^*$. Integrability of θ therefore means that for all $X \wedge Y \in \wedge^2 C^*$, $i_Y i_X d\theta = 0$. But this means that $d\theta$ must lie entirely in $\langle\theta\rangle \otimes C$. $\square$