MAT1312 Notes

October 7, 2015

Remark. *Recall the last class we saw the following:*

$$F \subset_{corank\ k} TM \text{ subbundle } \longleftrightarrow I =< \theta_1, ..., \theta_k >_{\theta_i \in \Omega^1, \bigwedge \theta_i \neq 0} \subset \Omega^{\cdot}(M) \longleftrightarrow < \theta_1 \wedge ... \wedge \theta_k >_{\Omega^{\cdot}} \subset \Omega^k$$

$$F \longleftrightarrow AnnF \longleftrightarrow detAnnF$$

$$Integrability\ Conditions: \ [F, F] \subset F \longleftrightarrow dI \subset I \longleftrightarrow d\Omega = \eta \wedge \Omega \quad \eta \in \Omega^1$$

Example. *Lie algebra*
$g = sl_2\mathbb{R} =< \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} >_{/R} =< e, f, g >$
Observe the following: $g = T_1 \to T_gG$ *by either* $(L_g)_*$ *or* $(R_g)_*$. *Using* L_g *we have an identification* $L : G \times g \to TG$. *Choosing* $F_1 \in T_1G$ *we can transport* $G \times F_1$ *to* $L(G \times F_1) = F \subset TG$.
Observing that $[F_1, F_1] \subset F \to [F, F] \subset F$ *and* $[e, f] = h,\ [h, e] = 2e$,
$[h, f] = -2f$ *we can choose* $F =< h, e >$
Strategy: Find $F_1 \subset sl_2\mathbb{R} \to F \subset TG \to F_1$ *is a 2D foliation invariant under left action*
If $\Gamma \subset G$ *is discrete, co-compact subgroup then* $X = \frac{G}{\Gamma}$ *inherits foliation*
On G *the foliation is by cosets of* $exp(h, e) = \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$
Note: Dual basis e^*, f^*, h^* *gives basis for left invariant 1-forms* $\subset \Omega^1(G) \to \theta_F = f^* \ \ d\theta = df^*$

$$df^*(a, b) = af^*(b) + bf^*(a) - f^*([a, b]) = -f^*([a, b])$$

$$df^*(h, f) = -2f^*(-2f) = 2 \to \ df^* = 2h^* \wedge f^* \to d\theta = \eta \wedge \theta \to \ d\theta \wedge \theta = 0$$

$$GV(F) = \eta \wedge d\eta = 4(h^* \wedge dh^*) = -4(h^* \wedge e^* \wedge f^*)$$

Remark. *Geometric Description*
Let $G = PSL_2\mathbb{R}$
Observe $Aut(\mathbb{H}) = PSL_2\mathbb{R} \subset PSL_2\mathcal{C} \subset Aut(\mathcal{P}^1)$ *(automorphisms of Riemann sphere)*
G *acts transitively and freely on* $S^1T\mathbb{H} = \{$*unit tagent vectors to* $\mathbb{H}\}$

$$PSL_2\mathbb{R} \cong S^1T\mathbb{H} \cong S^1 \times \mathbb{H}$$

Furthermore for $\Gamma \subset PSL_2\mathbb{R}$ *we have* $\frac{\mathbb{H}}{\Gamma} = \Sigma$ *hyperbolic* $2D$ *surface*

$$S^1T\Sigma \cong \frac{S^1T\mathbb{H}}{\Gamma} \cong \frac{PSL_2\mathbb{R}}{\Gamma} = X$$

obtain foliation on $S^1T\Sigma$ *for any hyper surface*
nonzero GV *invariant* $4hypvol(S^1T\Sigma) = 8\pi hypvol(\Sigma)$
Each leaf $\mathcal{L} \subset X$ *is labelled by ideal point* x

$$\begin{array}{ccccc} X & = & S^1T\Sigma & \supset & \mathcal{L}_X \\ {\scriptstyle S^1\text{ fibration}}\downarrow & & {\scriptstyle \mathbb{H}\text{ fibration}}\downarrow \pi & & \downarrow \\ \Sigma & & S^1 & \ni & x \end{array}$$

Note: $\pi^* d\theta = \theta_*$ defines the foliation

Remark. *Wobble Effect*
Choose Riemannian metric g *on* X^3, θ *of unit norm,* $d\theta = \eta \wedge \theta$ *where* $\eta \perp \theta$

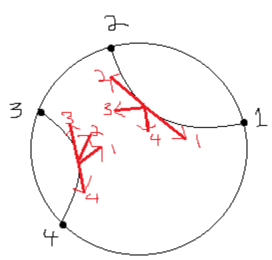
Theorem. *For any curve* $\gamma(t)$ *orthogonal to* $\mathcal{F}$ *foliation, let:*

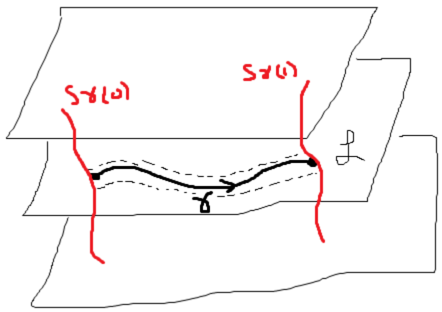
- κ *curvature*
- τ *torsion*
- N *normal vector*
- B *binormal vector*
- $\phi \in \Gamma(\mathcal{L}, S^2T^*)\mathcal{L} \otimes N\mathcal{L})$ $(u, v) \mapsto D_{N\mathcal{L}}(\nabla_u v)$ *second fundamental form*

Then

- $\eta \wedge d\eta = [\kappa^2(\tau + \phi(N, B))]vol_g$ *is the* GV *form*

  In our example above GV *form* $=\kappa^2\tau$





Remark. *Holonomy of a Foliation*
Let $\gamma : [0,1] \to \mathcal{L}$ *a path in a leaf* $\mathcal{L}$ *of folitaion* $\mathcal{F}$
Choose slices $S_{\gamma(0)}, S_{\gamma(1)}$ *emmbedded submanifolds of* M *transverse to foliations at* $\gamma(0), \gamma(1)$
of dim=cod$\mathcal{L} \to$ $T_{\gamma(i)}S_{\gamma(i)} \oplus T_{\gamma(i)}\mathcal{L} = T_{\gamma(i)}M$
The foliation defines a germ of diffeomorphisms $(S_{\gamma(0)}, \gamma(0)) \to (S_{\gamma(1)}, \gamma(1))$
In particular

$$\begin{array}{ccc} T_{\gamma(0)}S_{\gamma(0)} & \overset{\cong}{\to} & T_{\gamma(1)}S_{\gamma(1)} \\ \| & & \\ N_{\gamma(0)}\mathcal{L} & \overset{P_\gamma}{\to} & N_{\gamma(1)}\mathcal{L} \end{array}$$

"Parallel transport map" $P_\gamma : N_{\gamma(0)}\mathcal{L} \to N_{\gamma(1)}\mathcal{L}$
Identification P_γ *is the parallel transport of a partial flat connection* ∇ *on* $N\mathcal{F} = \frac{TM}{\mathcal{F}}$

Remark. *Partial Connections*
Let $F \subset TM$ *subbundle,* V *vector bundle*
Idea*: Should use* ∇ *to differentiate section* $s \in \Gamma(V)$ *along vector field* $y \in \Gamma(F)$ *resulting in* $\nabla_y s \in \Gamma(V)$

Definition. *A* ***partial connection*** *on* V *along* F *is* $\nabla : \Gamma^{/infty}(M, V) \to \Gamma^\infty(M, F^* \otimes V)$ $\mathbb{R}$*-linear s.t.*

- $\nabla(fs) = f\nabla s + (a^*df) \otimes s$ *where* $a : F \hookrightarrow TM$ *inclusion* $a^*df = df|_F$

Note: If $F = T$ *then this is equivalent to the usual connection*
If F *involutive then* ∇ *has a curvature tensor* $R^\nabla \in \Gamma^\infty(\bigwedge^2 F^* \otimes EndV)$

$$R^\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in End(V)$$

Note: If $F = T$ *then this is equivalent to the usual connection and curvature*

Definition. *The **Bott connection** is a partial connection* ∇ *on NF along F,* $\nabla : \Gamma(N) \to \Gamma(F^* \otimes N)$ *defined as follows:*
Let $Y \in \Gamma(F),\ v \in \Gamma(N),\ \pi : TM \to \frac{TM}{F}\ \tilde{v}$ *s.t.* $\pi\tilde{v} = v$

$$\nabla_Y v := \pi[Y, \tilde{v}]$$

Proposition. *Bott connection is well-defined*

Proof. Let $\tilde{v}'$ be an alternative to $\tilde{v}$. Want $\pi[Y, \tilde{v} - \tilde{v}'] = 0$

$$0 \to F \to TM \to N \qquad \tilde{v} - \tilde{v}' \to \tilde{v} - \tilde{v}' \textit{section of } F$$

$$\to [Y, \tilde{v} - \tilde{v}'] \in F \to \pi[Y, \tilde{v} - \tilde{v}'] = 0$$

□

Remark.

$$R^{\nabla}(X,Y)v = (\nabla_X \nabla_Y - \nabla_Y \nabla_X)v - \nabla_{[X,Y]}v = \pi([X, \widetilde{\pi[Y,\tilde{v}]}] - [Y, \widetilde{\pi[X,\tilde{v}]}] - [[X,Y],\tilde{v}])$$

Choosing the obvious extention $[Y, \tilde{v}]$*:*

$$R^{\nabla}(X,Y)v = \pi([X,[Y,\tilde{v}]] - [Y,[X,\tilde{v}]] - [[X,Y],\tilde{v}]) = 0$$

This is zero by the Jacobi identity $\to$ *Bott connection is flat*
$N \to M \hookleftarrow \mathcal{L} \leftarrow (i^*N, i^*\nabla)$ *can pull* (N, ∇) *back to leaf* $\mathcal{L} \to$ *gives flat connection on* $N\mathcal{L}$
Parallel transport of this flat connection on $N\mathcal{L}$ *gives* $P_[\gamma] : N_{\gamma(0)} \to N_{\gamma(1)}$

Remark. *Foliated deRham Theory*
A $F \subset TM$ *involutive behaves like the tangent bundle, so we can define the deRham complex by:*

$$\Omega_F^k = \Gamma^\infty(M, \bigwedge^k F^*)$$

$$d_F : \Omega_F^k \to \Omega_F^k$$

$$d_F(\rho)(x_1, ..., x_k) = \Sigma_i (-1)^i x_i(\rho(x_1, ..., \hat{x}_i, ..., x_k)) + \Sigma_{i<j}\rho([x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_n)$$

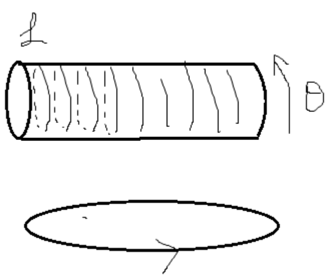
This yields a differential graded algebra $(\Omega_F^\bullet, \wedge, d_F)$

Remark. *Symbol Sequence*

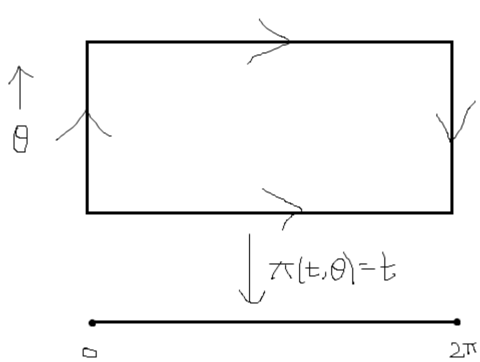
$$0 \quad \to \quad \Lambda^0 F^* \quad \xrightarrow{\sigma(\xi)} \quad \Lambda^1 F^* \quad \to \quad \cdots \quad \xrightarrow{\sigma(\xi)} \quad \Lambda^{rnkF} F^* \quad \to \quad 0$$

$\sigma(D) \in TM \otimes Hom(E,F) \qquad D : \Gamma(E) \to \Gamma(F)$
$\sigma(\xi) = a^*\xi \wedge - : \bigwedge^k F^* \to \bigwedge^{k+1} F^*$ *fails to be ellitpic complex* $\to$ *should not expect finite dimension cohomology*

Example. *1*
$\pi : X = S^1 \times S^1 \to S^1 \qquad \pi = proj_1 \qquad F = ker\pi_* = vertical\ direction$
$\Omega^0_F = C^\infty(X, \mathbb{R}) = \{f(\theta, t)\ periodic\}$
$\Omega^1_F = \Gamma(X, F^*) \cong C^\infty(X)d\theta = \{g(\theta, t) periodic\}$
$d_F(f \in \Omega^0_F) = (\partial_\theta f)d\theta \qquad df = 0 \leftrightarrow f = f(t)$
$H^0_F = \Omega^0(S)$ *infinite dimensional vector space (constant from the point of view of foliation)*
$H^1_F = \frac{all\ F^*\ 1-forms}{exact\ 1-forms} = \{g(t)d\theta\} = \Omega^0 d\theta$ *infinite dimensional vector space*



Example. *2. Klein Bottle*
$\mathbb{R}_t \times \mathbb{R}_\theta \ \ (t, \theta)\ (t + 2\pi, -theta)$ *bundle of* S^1 *over* S^1
$F =< \partial\theta >$ *vertical foliation with leaves* $\cong S^1 \to$ *nontrivial* S^1
$H^0_F = \Omega^0(S)$
$H^1_F = \{g(t)d\theta : g(t + 2\pi) = g(-t)\}$



Remark. *In these cases* $\pi : X \to S$ *foliation = level sets of submersion to S*
Observe that if H^k_F *is a module over* $C^\infty(S, \mathbb{R})$ *and if fibres* $\pi : X \to S$ *(leaves) are compact manifolds then* H^k_F *is a section of* VB *of* S *whose fiber at* $s \in S$ *is* $H^k(\pi^{-1}(s))$ *=finite dimensional vector space*