Part of the assignment is to understand the language used in the statement of the problems: work together and ask me questions after class. With the exception of the bonus question, I am not asking for any hard labour; you can give minimalistic arguments.

Exercise 1. $\mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{C}^2$ via $(\lambda_1, \lambda_2) \cdot (x, y) = (\lambda_1 x, \lambda_2 y)$. We obtain $\mathbb{C}P^1$ by taking the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the diagonal subgroup $\mathbb{C}^*$. The residual $\mathbb{C}^*$ action on $\mathbb{C}P^1$ is then given by

$$\lambda \cdot [x : y] = [\lambda x : y] = [x : \lambda^{-1} y]. \tag{1}$$

Describe the orbits of the action, as well as the relation of containment of orbit closures. Contrast this with the orbits for the induced action by the compact group $U(1) \subset \mathbb{C}^*$.

Now consider the $\mathcal{O}(1)$ line bundle; recall that it can be described as a quotient of $(\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}$ by the $\mathbb{C}^*$ action

$$\mu \cdot ((x, y), z) = ((\mu x, \mu y), \mu z). \tag{2}$$

a) Extend the $\mathbb{C}^*$ action on $\mathbb{C}P^1$ to an action on the total space of the $\mathcal{O}(1)$ line bundle, in such a way that the action is by vector bndle morphisms. Describe any choices involved.

b) Describe the orbit structure and orbit closure relations for $\mathcal{O}(1)$ and how these depend on the possible choices above.

Exercise 2. As in class, define the line bundle $\mathcal{O}(n)$ over $\mathbb{P}(V)$ by exhibiting its total space as a quotient of $V \setminus \{0\} \times \mathbb{C}$ by the $\mathbb{C}^*$-action

$$\lambda \cdot (v, z) = (\lambda v, \lambda^n z). \tag{3}$$

Give trivializations of $\mathcal{O}(n)$ over the affine charts given by the nonvanishing sets of a basis of sections of $\mathcal{O}(1)$, and compute transition functions for the line bundle $\mathcal{O}(n)$. Prove that $\mathcal{O}(m) \otimes \mathcal{O}(n) = \mathcal{O}(m+n)$ as line bundles.

Exercise 3. Give an argument explaining the fact that the tangent space to $\mathbb{P}(V)$ at the point $[v]$, $v \in V \setminus \{0\}$ can be naturally identified with the space of linear maps from $[v]$ to $V/[v]$. Using the Euler exact sequence which defined the tautological line bundle, namely

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \underline{V} \longrightarrow Q \longrightarrow 0\ , \tag{4}$$

show that the tangent bundle to $\mathbb{C}P^1$ is isomorphic to $\mathcal{O}(2)$. Show furthermore that the determinant line bundle of $T\mathbb{P}(V)$ is in general $\mathcal{O}(\dim V)$.

Exercise 4. The group $PGL(2, \mathbb{C})$ is the quotient of $GL(2, \mathbb{C})$ by its center. Show that $PGL(2)$ is an affine variety.

Exercise 5. Prove that the homogeneous ideal cutting out the pair of points $\{[1{:}0{:}0], [0{:}1{:}0]\}$ in $\mathbb{C}P^2$ is $(X_2, X_0 X_1)$. Describe geometrically why this is the case.

Exercise 6. Let $U = \mathbb{C}^{n+1}$ and $V = \mathbb{C}^{m+1}$. The projective space $\mathbb{P}(\mathrm{Hom}(U, V))$ of projective matrices has various natural subsets such as the set of matrices of rank $\leq k$. These subsets are cut out by homogeneous ideals generated by the minors of size $k + 1$.

a) Show that in the case $n = m = 1$, the rank ≤ 1 locus is a smooth projective quadric in 3 dimensions.

b) Show that in this case the quadric may be identified with $\mathbb{P}^1 \times \mathbb{P}^1$ via the map $\mathbb{P}(U^*) \times \mathbb{P}(V) \to \mathbb{P}(\mathrm{Hom}(U, V))$ defined by

$$([\xi], [v]) \mapsto [\xi \otimes v]. \tag{5}$$

c) Show that the curves $\mathbb{P}^1 \times \{q\}$ and $\{p\} \times \mathbb{P}^1$ in the domain are mapped to projective lines in the codomain, defining two rulings of the quadric by lines.

This is a special case of the *Segre embedding* of $\mathbb{P}(U) \times \mathbb{P}(V)$ in $\mathbb{P}(U \otimes V)$ via

$$([u], [v]) \mapsto [u \otimes v]. \tag{6}$$

d) Write the Segre embedding in terms of homogeneous coordinates on the domain and codomain.

e) Describe a homogeneous ideal of the codomain which cuts out the image of the Segre embedding, known as the *Segre variety*.

f) Explain which line bundle and linear system on the domain effectuates the Segre embedding.

Exercise 7. The *Veronese embedding* of degree d is the map $\mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^d(V))$ defined by

$$[v] \mapsto [v^d = v \otimes \cdots \otimes v]. \tag{7}$$

The image is called the *Veronese variety.* In the case $\mathbb{P}(V) = \mathbb{C}P^1$, the Veronese variety is called the rational normal curve of degree d.

a) In coordinates $[t_0 : t_1] \in \mathbb{P}^1$ and $[X_0 : \ldots : X_d] \in \mathbb{P}^d$, describe explicitly the embedding of $\mathbb{P}^1$ as the rational normal curve.

b) Show the curve has degree d by counting its intersection points with a generic projective hyperplane.

c) Which linear system on $\mathbb{C}P^1$ effectuates the embedding as a rational normal curve?

The rational normal curve of degree 2 is a smooth conic in $\mathbb{P}^2$. In the degree 3 case, it is called a *twisted cubic* curve in $\mathbb{P}^3$.

Bonus: Show that the homogeneous ideal corresponding to the twisted cubic is the one defined by the 2×2 minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \end{pmatrix} \tag{8}$$

To show that these three generators of degree 2 are contained in the ideal is easy; it is more difficult to show that any homogeneous polynomial F in the ideal must be a combination of the above three generators.