Exercise 1. Read section 1.3 and, after finishing the assignment, section 1.4 in Schmitt.

Exercise 2. Let $G = \mathbb{C}^*$ act on $V = \mathbb{C}^3$ via $\lambda \mapsto \mathrm{diag}(\lambda^{-a}, \lambda^b, \lambda^c) \in \mathrm{GL}_3(\mathbb{C})$, where $a, b, c \in \mathbb{Z}_{>0}$.

1. Describe the invariant ring $\mathbb{C}[V /\!\!/ G]$ for $a = b = 1$ and $c = 2$, and describe the categorical quotient variety.

2. Under what conditions on a, b, c is the invariant ring a polynomial ring?

Exercise 3. Let G be a reductive linear algebraic group and let $K \subset G$ be a compact real form. If G is acting on a finite-dimensional vector space V, with action map $\rho : K \times V \to V$, prove that

$$f \mapsto \int_K (\rho^* f) dv_K$$

defines a Reynolds operator $\mathbb{C}[V] \to \mathbb{C}[V]^G$. Try also to prove it without using Zariski density – use the polar decomposition of G.

Exercise 4. Suppose that G is affine algebraic group acting on affine variety V. Suppose that the categorical quotient $V /\!\!/ G$ exists. Show that $\mathbb{C}[V]^G$ must be finitely generated.

Exercise 5. Suppose that G is a reductive affine algebraic group acting on an affine variety V with categorical quotient $\pi : V \to V /\!\!/ G$. Show that if $W \subset V$ is a closed G-invariant subvariety, then $\pi(W)$ is a closed subvariety, and defines the categorical quotient of the induced G-action on W.

Exercise 6. Let G be an affine algebraic group acting on an affine algebraic variety V. Prove that each orbit $\mathcal{O}_v$, $v \in V$, is open in the closed variety $\overline{\mathcal{O}_v} \subset V$.

Exercise 7. Let $GL_n(\mathbb{C})$ act on $\mathrm{End}(\mathbb{C}^n)$ by the adjoint action. Prove that the ring of invariants is a polynomial ring generated by the coefficients of the characteristic polynomial. Prove that an alternative generating set consists of the functions given by

$$\mathrm{End}(\mathbb{C}^n) \ni A \mapsto \mathrm{Tr}^k(A) \in \mathbb{C}.$$

Describe the closed orbits parametrized by this categorical quotient.