GIT, Comments and questions

1. (Topology and Hilbert-Mumford) There was a question about what topology is used in the limit statement in the traditional statement of the Hilbert-Mumford criterion. Clearly we cannot interpret this as a statement about actual limits in the Zariski topology. Actually, the Hilbert-Mumford criterion is, in the ideal treatment, used for a reductive group action on a projective variety, where it is possible to extend any regular map from $\mathbb{C}^*$ to a map from $\mathbb{C}P^1$ (c.f. the valuative criterion for properness). In this case there is a clear meaning to the limit condition: it is simply an evaluation at the point at infinity in the $\mathbb{C}P^1$.

   In the case where G is acting on an affine variety, we may use the analytic topology and think of limits, or we may find a way of embedding the affine GIT quotient procedure into a projective one and use the procedure above, thereby avoiding the use of the analytic topology. We will see how this works later.

2. (Behaviour of H-M limits) Suppose that we have a linear representation V of G and that λ is a 1-PS for which

$$\lim_{z \to \infty} \lambda(z) \cdot v = w$$

   for $v, w \in V$. When this happens, it may be that w is not in the same G-orbit as v. For example, for the $\mathbb{C}^*$ action on $\mathbb{C}^2$ given by $\mathrm{diag}(z^{-1}, 1)$, we have

$$\lim_{z \to \infty} \lambda(z) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

   and $(0, 1)$ is not in the same $\mathbb{C}^*$-orbit as $(1, 1)$.

   However, such a limit point w could easily be in the orbit of v for the G-action: if we view the example above as a 1-PS of the $GL_2(\mathbb{C})$-action on $\mathbb{C}^2$, we see that the points are in the same orbit.

   Nevertheless, the H-M criterion says that v is not stable, and this is clear because w is stabilized by the 1-PS λ, and so if w were in the same orbit as v, it would not have the same dimension as G.

3. (H-M and maximal tori) There was a question about how many 1-PS we actually need to check in practice. The example of binary quantics showed that checking a single 1-PS precludes highly degenerate roots at only one point in P^1, whereas the condition must hold at all points. So in this sense, we must apply H-M to all 1-PS.

4. Algebraic group acting on affine variety with an orbit whose closure contains more than one closed orbit: consider the group of affine transformations of the plane (semidirect product of $\mathbb{C}^*$ with $\mathbb{C}$) acting via

$$\begin{pmatrix} 1 & \lambda \\ 0 & \mu \end{pmatrix}$$

   The open dense orbit has closure containing infinitely many closed orbits.