1 Extensions

Let U, W be finite-dimensional vector spaces. An *extension* of W by U is a vector space V, together with linear maps α, β such that

$$0 \longrightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta} W \longrightarrow 0 \tag{1}$$

is an exact sequence of linear maps, meaning that the kernel of each map is equal to the image of the previous map in the sequence.

Exercise 1. A *splitting* s of the sequence (1) is a map $s : W \to V$ such that $\beta s = \mathbf{1}_W$.

a) Prove that such a splitting exists.

b) Prove that a splitting s induces a map $t : V \to U$ such that $s\beta + \alpha t = \mathbf{1}_V$. As a result, show that $t\alpha = \mathbf{1}_U$ and that this gives an isomorphism between V and $U \oplus W$.

c) Prove that the set of splittings of (1) is an affine space modeled on $\mathrm{Hom}(W, U)$.

Recall that an affine space A modeled on a vector space V is a set A of "points" together with a transitive and faithful action of the group $(V, +)$ of "translations". Essentially, A is a copy of V without a distinguished origin.

Exercise 2. Let X, Y be finite-dimensional vector spaces and let $R \in \mathrm{Hom}(X, Y)$. As described in class, this gives rise to two exact sequences

$$0 \longrightarrow \mathrm{Ker}\, R \longrightarrow X \longrightarrow \mathrm{Im}\, R \longrightarrow 0 \tag{2}$$

$$0 \longrightarrow \mathrm{Im}\, R \longrightarrow Y \longrightarrow \mathrm{Cok}\, R \longrightarrow 0\ , \tag{3}$$

where $\mathrm{Cok}\, R$ is the cokernel of R, defined by the quotient $Y/\mathrm{Im}\, R$. Prove that if bases for $\mathrm{Ker}\, R$, $\mathrm{Im}\, R$, and $\mathrm{Cok}\, R$ are chosen, and if splittings of (2) and (3) are chosen, then this defines bases for X and Y, and that relative to these bases, R has a matrix with a block form given by

$$R = \begin{pmatrix} \mathbf{1}_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}, \tag{4}$$

where r is the rank of R and $\mathbf{1}_{r\times r}$ is an identity matrix of size r.

Exercise 3. Let (V, α, β) and (V', α', β') be two extensions of W by U. A morphism of extensions from V to V' is a linear map $\psi : V \to V'$ rendering the following diagram commutative.

$$\begin{array}{ccccc} & & V & & \\ & \overset{\alpha}{\nearrow} & \downarrow \psi & \overset{\beta}{\searrow} & \\ U & & & & W \\ & \underset{\alpha'}{\searrow} & & \underset{\beta'}{\nearrow} & \\ & & V' & & \end{array} \tag{5}$$

a) Prove that any such morphism is an isomorphism.

b) Prove that any two extensions of W by U are isomorphic.

c) Choose bases for W and U, and concatenate to form a basis for the trivial extension $V = U \oplus W$. Describe the matrix of a general automorphism of the extension V.

2 Koszul sequences

Let V be a vector space of dimension n, and fix $\xi \in V^* \setminus \{0\}$. Then ξ defines an operator e_ξ on $\wedge^\bullet V^*$ defined by

$$e_\xi(\sigma) = \xi \wedge \sigma. \tag{6}$$

Note that e_ξ sends $\wedge^k V^*$ to $\wedge^{k+1}V^*$; in other words it is an operator of degree $+1$.

Now choose $v \in V$ and define an operator i_v on $\wedge^\bullet V^*$ as follows: on $\wedge^0 V^*$ it is zero, and on $\wedge^1 V^*$ it is given by the pairing between V and V^*, so that for $\alpha \in \wedge^1 V^* = V^*$, we have

$$i_v\alpha = \alpha(v). \tag{7}$$

To complete the definition of i_v, we require that it is a graded derivation of $\wedge^\bullet V^*$, which means that it satisfies the following Leibniz rule:

$$i_v(\alpha \wedge \beta) = (i_v\alpha) \wedge \beta + (-1)^\alpha \alpha \wedge i_v\beta. \tag{8}$$

The operator defined in this way is called "the interior product by v" and is of degree -1.

Exercise 4.

a) Prove that the following sequence is an exact sequence:

$$0 \longrightarrow \wedge^0 V^* \xrightarrow{e_\xi} \wedge^1 V^* \xrightarrow{e_\xi} \cdots \xrightarrow{e_\xi} \wedge^{n-1} V^* \xrightarrow{e_\xi} \wedge^n V^* \longrightarrow 0\ . \tag{9}$$

b) Prove that $i_v i_v = 0$.

c) Prove that the following sequence is an exact sequence:

$$0 \longleftarrow \wedge^0 V^* \xleftarrow[i_v]{} \wedge^1 V^* \xleftarrow[i_v]{} \cdots \xleftarrow[i_v]{} \wedge^{n-1} V^* \xleftarrow[i_v]{} \wedge^n V^* \longleftarrow 0\ . \tag{10}$$

d) Compute the operator $i_v e_\xi + e_\xi i_v$. It should be an operator of degree 0, of course.

e) Fix ξ and v as above, but with $\xi(v) = 1$. Prove that any k-form $\alpha \in \wedge^k V^*$ can be decomposed as follows:

$$\alpha = \xi \wedge \beta + \gamma, \tag{11}$$

for a unique pair (β, γ) such that $i_v\beta = i_v\gamma = 0$.